Numerical Analysis Preliminary Examination
Spring 2020

Problem 1. Consider a matrix $A \in \mathbb{C}^{n\times m}$. The norm $||A||_F$ is given by

$$||A||_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m}|A_{i,j}|^2}.$$

Show that

$$||A||_F = \sqrt{\sum_{k=1}^{\min\{n,m\}}\sigma_k^2}$$

where σ_k, $k = 1, 2, \ldots, \min\{n, m\}$ are the singular values of the matrix A.

Problem 2. Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite and $b \in \mathbb{R}^n$. Consider the conjugate gradient method for finding the unique solution x^* to $Ax = b$. Let $x_0 \in \mathbb{R}^n$ be the initial vector used in the method and let $r_0 = b - Ax_0$. Suppose the method is carried out to infinite precision.

(a) Prove the method finds x^* in one step (in other words, $x_1 = x^*$) if and only if r_0 is either the zero vector or an eigenvector of A.

(b) Suppose A has $m \leq n$ distinct eigenvalues. Show the method finds x^* in at most m iterations.

Problem 3. Let $A = A_{ij} \in \mathbb{C}^{m\times n}$ be a complex-valued matrix. The matrix $X \in \mathbb{C}^{n\times m}$ is said to satisfy the Moore-Penrose equations (MP) if:

$$AXA = A$$
$$XAX = X$$
$$(AX)^* = AX$$
$$(XA)^* = XA$$

(where, for $B \in \mathbb{C}^{m\times n}$, B^* denotes the complex conjugate transpose, $B^* = \bar{B}^T$).

(a) Given a matrix A, show there is at most one matrix X that satisfies (MP).

(b) Suppose $A = V\Sigma W^*$ is a singular value decomposition of A. Define $A^\dagger$ as follows.

- For a complex scalar λ, let

$$\lambda^\dagger = \begin{cases} \frac{1}{\lambda} & \lambda \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

- If A is *diagonal* (i.e. $A_{ij} = 0$ whenever $i \neq j$) define $A^\dagger \in \mathbb{C}^{n\times m}$ by

$$(A^\dagger)_{ij} = (A_{ji})^\dagger.$$

- Otherwise, define

$$A^\dagger = W\Sigma^\dagger V^*.$$

Show that $A^\dagger$ is a solution to (MP).

(c) For $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$, show that $x = A^\dagger b \in \mathbb{C}^n$ is the vector of minimum norm that minimizes $||Ax - b||_2$.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 2 & 10^2 & 10^2 \\ 10^{-2} & 1 & 10^2 \\ 10^{-2} & 10^{-2} & 2 \end{bmatrix}.$$

Show that the eigenvalues of A lie in $D_\varepsilon(1) \cup D_\varepsilon(2)$ where $\varepsilon = 10^{2/3} + 10^{-2/3}$ and the notation $D_\varepsilon(x)$ should be interpreted as

$$D_\varepsilon(x) = \{\lambda \in \mathbb{C} : |\lambda - x| < \varepsilon\}.$$

Hint: You might want to use a matrix of the form $T = \operatorname{diag}\{1, \alpha, \alpha^2\}$ *with a carefully selected value of* α*.*