PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Spring 2019

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let Ω be a bounded domain (open, connected) in $\mathbb{R}^n$. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of

$$\Delta u + \sum_{k=1}^{n} a_k(x)\partial_{x_k} u + c(x)u = 0$$

where a_k, $c \in C(\overline{\Omega})$ and $c < 0$ in Ω. Show that if $u = 0$ on $\partial\Omega$, then $u = 0$ in all of Ω.

2. Solve the equation

$$\partial_t u + x\partial_x u + u = 0$$

with initial data $u(0, x) = f(x)$, where f is a compactly supported smooth function on $\mathbb{R}$. Sketch the graph of $u(t, x)$ as a function of x for a non-trivial compactly supported initial datum of your choice for $t = 0$, $t = R$ and $t = -R$ for R large (what happens to the support and amplitude?)

3. Suppose $u(t, x)$ is a function on $\mathbb{R} \times \mathbb{R}^3$ that satisfies the nonlinear wave equation

$$(\partial_t^2 - \Delta)u = u^3.$$

Assume that $u \in C^2(\mathbb{R} \times \mathbb{R}^3)$ and that $u(t, x)$ is compactly supported in x for each t. Define the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u(t, x)|^2 + (\partial_t u(t, x))^2)\, \mathrm{d}x.$$

(a) Prove that

$$\partial_t E(t) = \int_{\mathbb{R}^3} u(t, x)^3 \partial_t u(t, x)\, \mathrm{d}x, \quad \text{and that} \quad \partial_t E(t) \leq \|\partial_t u(t)\|_{L^2(\mathbb{R}^3)} \cdot \|u(t)\|^3_{L^6(\mathbb{R}^3)}.$$

(b) Prove that there exists a universal constant C independent of u such that

$$\partial_t E(t) \leq C E(t)^2.$$