1. Let $X_1, \ldots, X_n$ be an i.i.d. sample from $N(\theta, \theta)$ distribution, where $\theta > 0$.

   (a) Find a non-constant pivotal quantity (that is, a function of $X_1, \ldots, X_n$ and $\theta$ whose distribution does not depend on $\theta$). *Hint:* See if you can make $\sum_{i=1}^{n} X_i$ into a pivot via simple transformations involving $\theta$.

   (b) Use the pivotal quantity from part (a) to construct a confidence interval for $\theta$.

2. For nonnegative integers $\theta, n, N$ with $n \leq N$ and $\theta \leq N$, the hypergeometric distribution $\text{Hyper}(\theta, n, N)$ has p.d.f.

$$f_\theta(x) = P_\theta(X = x) = \binom{\theta}{x}\binom{N-\theta}{n-x} \Big/ \binom{N}{n} \tag{1}$$

   for nonnegative integers $x$ and values of the parameters such that the above quotient in (1) is defined, with $f_\theta(x) = 0$ otherwise. Recall that this is the distribution of the number of white balls in a simple random sample without replacement of size $n$ from an urn containing $\theta$ white balls and $N - \theta$ black balls. Throughout this problem we shall treat $n$ (i.e., the sample size) and $N$ (i.e., the population size) as fixed and known, and $\theta$ as an unknown parameter.

   (a) If $X \sim \text{Hyper}(\theta, n, N)$, show that this family of distributions has the monotone likelihood ratio property in $X$.

   (b) Given $\alpha \in (0, 1)$, give the form of an exactly level-$\alpha$ UMP test of $H_0 : \theta = N$ vs. $H_1 : \theta < N$ in as simple a form as possible, involving $X$. Your test may need to involve randomization to achieve the exact level $\alpha$. Justify that the test is UMP. *Hint:* For this you only need to consider $f_N(x)$, which takes a particularly simple form.

   (c) Suppose that among the urn's $N$ balls, only 1 is black. Given $\alpha, \beta \in (0, 1)$, find an expression for the smallest sample size $n$ guaranteeing that your level-$\alpha$ UMP test will reject $H_0$ with probability at least $1 - \beta$.