1. Let $X_1, \ldots, X_n$ be independent with distribution $\mathcal{U}[\theta-1, \theta]$, the uniform distribution on the interval $[\theta - 1, \theta]$, for some unknown $\theta \in \mathbb{R}$. Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ denote the order statistics.

   (a) Find, with explanation, an unbiased estimator $\widehat{\theta}$, which is a function only of $X_{(1)}$.

   (b) By finding, and proving the superiority (in mean square error) of another unbiased estimator $\widetilde{\theta}$ that dominates $\widehat{\theta}$ in (a), show that $\widehat{\theta}$ is not UMVU. (Hint: To avoid some cumbersome computation, you may use the fact that $X_{(1)}$ and $X_{(n)}$ are not linearly dependent, without proving it.)

   (c) Consider the following statement: Since $\widehat{\theta}$ is not UMVU, $X_{(1)}$ must either be not complete or not sufficient.

      i. From what theorem does the statement follow?

      ii. Give a direct proof that $X_{(1)}$ fails to be (your choice) either complete or sufficient.

2. (a) Let $\{\mathcal{P}_\theta,\ \theta \in \Theta\}$ be a family of probability distributions, and $X \sim \mathcal{P}_\theta$ for some $\theta \in \Theta$. Prove that if $T(X), T'(X)$ are both uniformly minimum-variance unbiased estimators (UMVUEs) of θ, then $T = T'$, $\mathcal{P}_\theta$-almost surely for every $\theta \in \Theta$.

   Hint: Letting $v = \mathrm{Var}(T) = \mathrm{Var}(T')$ denote the minimal variance and $\overline{T} = (T + T')/2$, compare $\mathrm{Var}(\overline{T})$ to v and use the Cauchy-Schwarz inequality $\mathrm{Cov}(T, T') \leq [\mathrm{Var}(T)\mathrm{Var}(T')]^{1/2}$ to show that $\mathrm{Cov}(T, T') = v$. Use this to show that $\mathrm{Var}(T - T') = 0$.

   (b) Let two independent sequences $X_1, \ldots, X_m$ i.i.d. $N(\mu, \gamma\sigma^2)$ and $Y_1, \ldots, Y_n$ i.i.d. $N(\mu, \sigma^2)$ be observed. Here, the X_i are independent of the Y_j, $\gamma > 0$ is known, and μ and $\sigma^2 > 0$ are unknown. Find the UMVUE of μ, and prove that it is UMVUE.

   (c) Let two independent sequences $X_1, \ldots, X_m$ be i.i.d. $N(\mu, \sigma_x^2)$ and $Y_1, \ldots, Y_n$ i.i.d. $N(\mu, \sigma_y^2)$ be observed. Here, the X_i are independent of the Y_j, μ is unknown, and $\sigma_x^2 > 0$ and $\sigma_y^2 > 0$ are unknown and not assumed to be equal. Show that the UMVUE of μ does not exist.

   Hint: Let $\gamma = \sigma_x^2/\sigma_y^2$ and apply part (b) to say what the UMVUE is. Then reverse the roles of the X's and Y's, repeat, and finally apply part (a) for a contradiction.