Geometry and Topology Graduate Exam
Spring 2019

Solve all 6 problems. Partial credit will be given to partial solutions.

Problem 1. Let $X = S^2/\sim$ be the quotient of the sphere

$$S^2 = \big\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\big\}$$

by the equivalence relation $\sim$ that glues together the three points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$; namely, one equivalence class of $\sim$ is equal to $\big\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\big\}$, and all other equivalence classes consist of single points. Compute the fundamental group $\pi_1(X; x_0)$ for your preferred choice of base point $x_0 \in X$.

Problem 2.

Recall that the wedge sum $Y \vee Z$ of two spaces Y and Z, each equipped with a base point y_0 and z_0, is obtained from the disjoint union $Y \coprod Z$ by gluing $x_0 \in X$ to $y_0 \in Y$. Let $X = S^1 \vee S^2$ be the wedge sum of the circle S^1 and the sphere S^2 (for arbitrary choices of base points).

- **a.** Draw a picture of the universal cover $\widetilde{X}$ of X.
- **b.** Compute the homology group $H_2(\widetilde{X}; \mathbb{Z})$, with integer coefficients.

Problem 3. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = 2z^3 + 3z^2$. Note that $f^{-1}\big(\{0, 1\}\big) = \{-\frac{3}{2}, -1, 0, \frac{1}{2}\}$ (no need to check this).

- **a.** Show that the restriction $g : \mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\} \to \mathbb{C} - \{0, 1\}$ of f is a covering map. Hint: first show that g is a local diffeomorphism.
- **b.** What is the index of the subgroup $g_*\big(\pi_1\big(\mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\}; 1\big)\big)$ in the fundamental group $\pi_1\big(\mathbb{C} - \{0, 1\}; 5\big)$?

Problem 4.

Let M be a smooth m-dimensional submanifold of $\mathbb{R}^n$, and let

$$S_r^{n-1} = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n; \sum_{i=1}^{n} x_i^2 = r^2\big\}$$

denote the sphere of radius r centered at the origin in $\mathbb{R}^n$. Show that, for every $\varepsilon > 0$, there exists an r in the interval $[1 - \varepsilon, 1 + \varepsilon]$ such that the intersection $M \cap S_r$ is a submanifold of M of dimension $m - 1$. Possible hint: consider the map $f : M \to \mathbb{R}$ defined by $f(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2$.

Problem 5.

Let $M = \{(x, y, z, w) \in \mathbb{R}^4; x^2 + y^2 + z^2 - w^4 = -1\}$.

- **a.** Prove that M is a differentiable submanifold of $\mathbb{R}^4$.
- **b.** Let f be the map $\mathbb{R}^4 \to \mathbb{R}$ sending $(x, y, z, w) \mapsto w$. Compute the critical values of the restriction $f_{|M} : M \to \mathbb{R}$. Possible hint: the tangent map $T_p f_{|M}$ of the restriction $f_{|M}$ at $p \in M$ is the restriction of $T_p f$ to $T_p M = \ker T_p g$ where $g : \mathbb{R}^4 \to \mathbb{R}$ is defined by $g(x, y, z, w) = x^2 + y^2 + z^2 - w^4$.

Problem 6. Let Z be the vector field on $\mathbb{R}^2$ defined by $Z(x, y) = -y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}$. Compute the Lie derivative $\mathcal{L}_X(dx \wedge dy)$.