REAL ANALYSIS

Spring 2019

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Find the limit

$$\lim_{j\to\infty}\int_{-1}^{1}\frac{1-e^{-\frac{x^2}{j}}}{x^2}dx.$$

2. Let F be absolutely continuous and $F, F' \in L^1(\mathbb{R}, m)$, where $m(dx) = dx$ is the Lebesgue measure. Prove that

$$\int_{-\infty}^{\infty} F'(x)\,dx = 0.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $h \circ f$ is integrable for every continuous function $h : \mathbb{R} \to \mathbb{R}$. Prove that there is $a > 0$ so that

$$\mu(\{x : |f(x)| > a\}) = 0.$$

4. (i) Show that for every $\varepsilon > 0$ there exists a non-negative $f \in L^1([0,1], m)$ such that $f(x) = 0$ on the set of measure $\geq 1-\varepsilon$ and

$$\int_a^b f(x)\,dx > 0$$

for all $0 < a < b < 1$.

(ii) Show that for each $\varepsilon > 0$ there exists an absolutely continuous strictly increasing h on $[0,1]$ so that $h'(x) = 0$ on a set of measure $\geq 1-\varepsilon$.