Algebra Qualifying Exam - Spring 2019

1. Classify all groups of order 495.
2. If $f(x) \in \mathbb{Q}[x]$ is irreducible, and has precisely 2 non-real complex roots, then if E is the splitting field of f over $\mathbb{Q}$, show that $Gal(E/\mathbb{Q})$ is isomorphic to S_5.
3. If R is a ring and M is a Noetherian (left) R-module, then prove any surjective R-module homomorphim $\varphi : M \to M$ is an isomorphism.
4. If R is an Artinian ring with no non-zero nilpotent elements, then show that R is a direct sum of division rings.
5. Let $R \subset S$ be an extension of commutative rings such that $S - R$ is closed under multiplication. Prove that R is integrally closed in S.
6. In the ring $\mathbb{C}[x, y, \frac{1}{y}]$, show that some power of $x^9 - 3y^3 + 4$ belongs to the ideal $(x^2 - y, 2x^2 - xy^2)$.