MATH 507a **QUALIFYING EXAM** **January 10, 2019**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

1. Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with density

$$f(x) = \begin{cases} |x|^{-3} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

and characteristic function $\phi(t)$. Let $S_n := X_1 + \cdots + X_n$.

(a) Show that $\mathbb{E}\,(X_1^2) = \infty$.

(b) It is a fact that $1 - \phi(t) \sim t^2 \log(1/|t|)$ as $t \to 0$; you are not asked to prove this. Using this fact, propose constants $a_1, a_2, \ldots$ and prove that S_n/a_n converges in distribution to the normal distribution with mean 0 and variance 1.

2. Let $X_1, X_2, \ldots$ be independent, uniformly distributed in $(0, 1)$. Let $S_n := X_1 + \cdots + X_n$, with $S_0 = 0$. For $t > 0$ let

$$N_t = \max\{n : S_n < t\}$$

so that $N_t < n$ iff $X_1 + \cdots + X_n \geq t$.
Show that N_t/t is almost surely convergent as $t \to \infty$, and find the limit.

3. Let $X_1, X_2, \ldots$ be i.i.d. exponential with mean 1, and let $Z_1, Z_2, \ldots$ be i.i.d. standard normal $N(0, 1)$. Recall that as $t \to \infty$,

$$\mathbb{P}(Z_1 > t) \sim \frac{1}{\sqrt{2\pi}}\,\frac{1}{t}\,e^{-t^2/2},$$

where $\sim$ means the ratio converges to 1.

(a) Find a sequence $\{t_n\}$ so that for all $b \in \mathbb{R}$, as $n \to \infty$, $\mathbb{P}(\max_{i \leq n} X_i \leq t_n + b)$ has a limit $F(b) \in (0, 1)$.

(b) Find a sequence $s_n \to 0$ so that $\mathbb{P}(\max_{i \leq n} Z_i \leq \sqrt{2 \log n} + s_n)$ has a limit in $(0, 1)$.

(c) Show that $\mathbb{P}(\max_{i \leq n} X_i > \max_{i \leq n} Z_i) \to 1$ as $n \to \infty$.