Numerical Analysis Prelim Spring 2019

January 11, 2019

Problem 1.
Let $\{\varphi_i\}_{i=1}^m$ be m linearly independent vectors in $\mathbb{R}^n$ and set $\Phi = [\varphi_1|\varphi_2|...|\varphi_m] \in \mathbb{R}^{n\times m}$.

(a) Prove that $\Phi^T\Phi$ is nonsingular

(b) Let $||\cdot||$ denote the Euclidean norm on $\mathbb{R}^n$, for $x \in \mathbb{R}^n$ define the map $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by $P_\Phi x = \varphi$, where $\varphi = \arg\min_{\psi\in\text{span}\{\varphi_i\}_{i=1}^m} ||x - \psi||$, and show that $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is linear.

(c) A linear transformation P on a Hilbert space is said to be an orthogonal projection if 1) P is self adjoint (i.e. $P^* = P$) and 2) P is idempotent (i.e. $P^2 = P$). Show that $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal projection.

(d) Find a set of vectors in $\mathbb{R}^n$ whose span is equal to the orthogonal complement of the subspace spanned by the $\{\varphi_i\}_{i=1}^m$.

Problem 2. For a given small value $\epsilon > 0$ consider a matrix A of the form

$$A = \begin{pmatrix} 1 & 1 \\ -\epsilon & \epsilon \end{pmatrix}.$$

a. Find the matrix operator norm $\|A\|_2$ induced by the Euclidean norm in $\mathbb{R}^2$. (Hint:$\|A\|_2 = \|Ax^*\|_2$ where $\|x^*\|_2 = 1$ and $\|Ax^*\|_2^2 \geq \|Ax\|_2^2$ for all $x \in \mathbb{R}^2$ with $\|x\|_2 = 1$.)

b. Find the matrix norm $\|A^{-1}\|_2$.

c. What is the smallest possible norm $\|\delta A\|_2$ for a matrix δA such that the matrix $A+\delta A$ is singular?

d. Find a matrix δA with the smallest possible norm $\|\delta A\|$ such that $A + \delta A$ is singular.

Problem 3. Consider a strictly diagonally dominant matrix $A = D - E - F$ where matrices D, E, F are diagonal, strictly lower-triangular, strictly upper-triangular matrices, respectively.

a. For any parameter $0 < \omega < 1$ and complex value λ with $|\lambda| \geq 1$, show that the matrix

$$A_{\lambda,\omega} = \omega^{-1}(1 - \omega - \lambda)D + F + \lambda E$$

has the same properties as matrix A; that is, that $A_{\lambda,\omega}$ is also a strictly diagonally dominant matrix. (Hint: Use the fact that $|1-\omega-\lambda| \geq |\lambda| - (1-\omega)$ to show that $|\omega^{-1}(1-\omega-\lambda)| \geq |\lambda|$)

b. Recall that the iteration matrix B_{SOR} for the Successive Over Relaxation (SOR) method is given by

$$B_{SOR} = (\omega^{-1}D - E)^{-1}((\omega^{-1}-1)D + F).$$

Show that the matrix $B_{SOR} - \lambda I$ is nonsingular for all $|\lambda| \geq 1$ and $0 < \omega < 1$.

c. Using the conclusion of part b above, what can be deduced about the convergence of the SOR method applied to a strictly diagonally dominant matrix A with $0 < \omega < 1$. Justify your answer.

Problem 4.

Consider the matrix

$$A = \begin{bmatrix} -2 & 10 & 100 & 200 \\ 0.01 & 5 & 100 & 1000 \\ .001 & .02 & 15 & 10 \\ 0 & 0 & .01 & 9 \end{bmatrix}$$

(a) Notice that A is close to upper triangular. From this observation alone, what do you expect to be true about its eigenvalues?

(b) Use Gerschgorin's theorem to locate the eigenvalues of A to within a region of the complex plane that is the union of four discs. Notice, the theorem does not do a very good job at locating the eigenvalues.

(c) Suppose T is invertible. Prove that the eigenvalues of $T^{-1}AT$ are the same as the eigenvalues of A.

(d) Find a diagonal matrix T such that when Gerschgorin's theorem is applied to $T^{-1}AT$, one obtains four disjoint disks in the complex plane, each of which contains an eigenvalue of A.