PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2018

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with normal vector field ν and let $u_0 \in C_b(\Omega)$ with $u_0 \geq 0$ be non-trivial. Show that the problem

$$\begin{aligned} \partial_t u - \Delta u &= u^2 \text{ in } \Omega \times (0,T), \\ \partial_\nu u &= 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \text{ for } x \in \Omega, \end{aligned}$$

   exists for at most a finite time T.

   Hint: Show that the mean $m(t) = \frac{1}{|\Omega|}\int_\Omega u(t,x)dx$ satisfies $\partial_t m(t) \geq m(t)^2$.

2. Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded and let $R > 0$ such that $\Omega \subset B_R(0)$.

   (a) Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $\Delta v = 0$ in Ω. Show that

   $$\max_{x \in \overline{\Omega}} v(x) = \max_{x \in \partial\Omega} v(x).$$

   (b) Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

   $$\begin{aligned} -\Delta u &= 1 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

   Show that

   $$0 \leq u(x) \leq \frac{R^2 - |x|^2}{2n}$$

   for all $x \in \overline{\Omega}$.

3. Let u be a classical solution of the following initial boundary value problem:

$$\begin{aligned} u_t &= u_{xx}, \qquad \text{in } (a,b) \times (0,T) \\ u(a,t) &= u(b,t) = 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

   where u_0 is a continuous function.

   (a) Show that the solutions are unique.

   (b) Show that there exists a constant $\alpha > 0$ such that

   $$\|u(\cdot,t)\|_{L^2}^2 \leq e^{-\alpha t}\|u_0\|_{L^2}^2.$$