1. Let $M(n; p_1, \ldots, p_k)$ denote the multinomial distribution with n trials and cell probabilities $p_1, \ldots, p_k$. Now let $X = (x_1, x_2, x_3, x_4)$ have multinomial distribution

$$M(200; 1/2 + \theta/2, 1/4 - \theta/4, 1/4 - \theta/3, \theta/12)$$

for some $\theta \in (0, 3/4)$. Write down the E-step and M-step of the Expectation Maximization algorithm for estimating θ by assuming that there is actually data $(x_{11}, x_{12}, x_{21}, x_{22}, x_3, x_4)$ from 6 cells rather than 4, but that the first 4 cells are unobserved but $x_1 = x_{11} + x_{12}$ and $x_2 = x_{21} + x_{22}$ are observed. *Hint:* Choose convenient cell probabilities for the unobserved cells in the complete model.

2. Recall that the bivariate normal distribution with mean $(\mu_1, \mu_2)^T$, variances σ_1^2, σ_2^2, and correlation coefficient ρ has density

$$f(y_1, y_2) \propto \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2 - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right\}\right] \tag{1}$$

for $y_1, y_2 \in (-\infty, \infty)$.

Now let θ be a random variable with (univariate) normal distribution $N(\mu, \tau^2)$ and, given θ, let $X_1, \ldots, X_n$ be i.i.d. with (univariate) normal distribution $N(\theta, \sigma^2)$. Let $\overline{X} = (1/n)\sum_{i=1}^n X_i$ denote the sample mean. For this problem assume that θ is unobserved, the X_i are observed, τ and σ are known values, and μ is an unknown parameter.

   (a) Find the joint distribution of θ and $\overline{X}$. Give the name of the distribution and the values of any parameters in terms of quantities defined above. *Hint:* It may help to use the change of variables $\widetilde{\theta} = \theta - \mu$ and $\widetilde{X} = \overline{X} - \mu$.

   (b) Using your answer from part (2a), find the marginal distribution of $\overline{X}$. Give the name of the distribution and the values of any parameters in terms of quantities defined above.

   (c) Using your answer from part (2b), for arbitrary $\alpha \in (0, 1)$ write down an exact $(1 - \alpha)$ confidence interval for μ in terms of observed random variables and known quantities. Compare its width with the standard interval for θ and comment.