1. Let $U_1, \ldots, U_n$ be iid with the uniform distribution $\mathcal{U}[\beta, \beta+1]$ for $\beta \in \mathbb{R}$ and $n \geq 2$, and let $U_{(1)} < \cdots < U_{(n)}$ be the order statistics of the sample.

   a. Show that $(U_{(1)}, U_{(n)})$ is sufficient but not complete.

   b. Show that $\overline{U}_n - 1/2$ is not a uniformly minimum variance estimator of β, where $\overline{U}_n$ is the average of $U_1, \ldots, U_n$.

2. Fix $n \geq 2$.

   (a) Let $y_{(1)} \leq \ldots \leq y_{(n)}$ be some ordered real numbers and define

   $$x_k = \sum_{i>k} y_{(i)} - (n-k) y_{(k)} \quad \text{for} \quad k = 1, \ldots, n, \tag{1}$$

   where an empty sum denotes 0 by convention. Show that x_k is a non-increasing sequence and $x_n = 0$.

   (b) Let $Y_1, \ldots, Y_n$ be independent with $Y_i \sim N(\theta_i, \sigma^2)$ for unknown $\sigma^2 > 0$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ in the $(n-1)$-simplex

   $$\theta_1 + \ldots + \theta_n = 1 \quad \text{and} \quad \theta_i \geq 0 \quad \text{for all } i. \tag{2}$$

   Based on the data $Y_1 = y_1, \ldots, Y_n = y_n$:

   i. find the MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$, subject to the constraints (2), by minimizing

   $$\frac{1}{2} \sum_i (y_i - \theta_i)^2 + \lambda \left(\sum_i \theta_i - 1 \right) \tag{3}$$

   over $\theta_i \geq 0$ and λ (Hint: Use part 2a, and consider the smallest $k \in \{1, \ldots, n\}$ such that $x_k < 1$, and $\lambda \in [y_{(k-1)}, y_{(k)})$.)

   ii. find the MLE of σ^2. You can state your answer in terms of $\widehat{\boldsymbol{\theta}}$ even if you don't solve part 2(b)i.