Geometry and Topology Graduate Exam
Spring 2018

Solve all **7** *problems. Partial credit will be given to partial solutions.*

Problem 1.

a) Define $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ by
$$F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 + x_3^2 + x_4^2, x_1^2 + x_2^2 - x_3^2 - x_4^2).$$
Show that $M = F^{-1}(1, 0)$ is a smooth manifold.

b) For each $x = (x_1, x_2, x_3, x_4) \in M$ show that the tangent space T_xM is spanned by $(x_2, -x_1, 0, 0)$ and $(0, 0, x_4, -x_3)$.

c) Let $G\colon \mathbb{R}^4 \to \mathbb{R}$ be a smooth map and $g = G\mid_M$ be the restriction of G to M. Show that x is a critical point of g if and only if $\ker dF_x \subset \ker dG_x$.

d) If $G(x_1, x_2, x_3, x_4) = x_1 + x_3$ find the critical points of g.

Problem 2. Let X be a topological space. Let SX denote the suspsension of X, i.e. the space obtained from $X \times [0, 1]$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another point:

$$SX := X\times[0, 1]/\sim, \quad \text{where } \{(x, t) \sim (y, s) \text{ if } s = t = 0 \text{ or } s = t = 1 \text{ or } (x, t) = (y, s)\}$$

Determine the relationship between the homology of SX and X.

Problem 3. Consider $\mathbb{R}^3$ with the coordinates (x, y, z). Write down explicit formulas for the vector fields X and Y which represent the infinitesimal generators of rotation about the x and y axes respectively and compute their Lie bracket.

Problem 4.

Draw a based covering space of the figure eight with each of the following subgroups of $\mathbb{Z} * \mathbb{Z} = \langle a, b\rangle$ as its fundamental group. In each case determine whether or not the subgroup is normal.

(1) $\langle a^3, b, aba^{-1}, a^{-1}ba\rangle$
(2) $\langle a^2, b^2, aba, bab\rangle$

Problem 5.

a) Prove there does not exist a degree 1 map $S^2 \to T^2$.

b) Let $f : M^p \to N^p$ be any smooth map between two connected compact orientable manifolds of the same dimension p. Suppose there exists a regular value $y \in N$ with three pre-images $f^{-1}(y) = \{x_1, x_2, x_3\}$. Prove that f is necessarily surjective.

Problem 6.

Let $T = S^1 \times S^1$ and let $x, y \in T$ be two distinct points. Let Y be the quotient space obtained from $T \times \{1, 2\}$ by identifying the points $(x, 1)$ and $(x, 2)$ into a single point $\bar{x}$, and identifying the points $(y, 1)$ and $(y, 2)$ into a single point $\bar{y}$, distinct from $\bar{x}$. Compute $\pi_1(Y, \bar{x})$.

ADDITIONAL PROBLEM ON NEXT PAGE

Problem 7.

a) Compute the integral

$$\int_{S^2} 2xyzdx \wedge dy + (yz + xy^2)dx \wedge dz + xzdy \wedge dz,$$

where S^2 is oriented as the boundary of the unit ball $B^3 \subset \mathbb{R}^3$.

b) Show that the form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{(0,0)\}$ is closed but not exact.