REAL ANALYSIS GRADUATE EXAM

Spring 2018

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $-\infty < a < b < \infty$ and suppose $\mathcal{B}$ is a countable collection of closed subintervals of (a, b). Give the proof that there is a countable pairwise-disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that

$$\bigcup_{I \in \mathcal{B}} I \subset \bigcup_{I \in \mathcal{B}'} \widetilde{I},$$

where $\widetilde{I}$ denotes the 5-times enlargment of I; thus if $I = [x - \rho, x + \rho]$ then $\widetilde{I} = [x - 5\rho, x + 5\rho]$.

2. Assume that f is absolutely continuous on $[0, 1]$, and assume that $f' = g$ a.e., where g is a continuous function. Prove that f is continuously differentiable on $[0, 1]$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $A_1, A_2, \ldots, A_{50} \in \mathcal{M}$. Assume that almost every point in X belongs to at least 10 of these sets. Prove that at least one of the sets has measure greater than or equal to $1/5$.

4. Let $f : [0, \infty) \to \mathbb{R}$ be absolutely continuous on every closed subinterval of $[0, \infty)$ and

$$f(x) = f(0) + \int_0^x g(t)\, dt, \qquad \text{for } x \geq 0,$$

where $g \in \mathscr{L}^1([0, \infty))$. Show that

$$\int_0^\infty \frac{f(2x) - f(x)}{x}\, dx = (\log 2) \int_0^\infty g(t)\, dt.$$