Algebra Qualifying Exam - Spring 2018

1. Prove that a group of order 72 cannot be simple.

2. Say that a group G is uniquely p-divisible if the p-th power map sending $x \in G$ to x^p is bijective. Show that if G is a finitely generated uniquely p-divisible *abelian* group, then G is finite and has order coprime to p.

3. Let $\mathbb{Q}$ be the field of rational numbers and consider $f(x) = x^8 + x^4 + 1 \in \mathbb{Q}[x]$. Write E for a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = Gal(E/\mathbb{Q})$. Find $|E : \mathbb{Q}|$ and determine the Galois group G up to isomorphism. If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of G on Ω.

4. Show that a 10-dimensional $\mathbb{C}$-algebra necessarily contains a non-zero nilpotent element (hint: what can you say about the Jacobson radical of such an algebra?).

5. Consider the algebra $A := \mathbb{C}[M_n(\mathbb{C})]$ of polynomial functions on the ring of $n \times n$ complex matrices $M_n(\mathbb{C})$. Consider the polynomial functions defined by the formula $P_{ij}(X) := (X^n)_{ij}$. Let $I \subset A$ be the ideal defined by P_{ij}, $1 \leq i, j \leq n$. Describe the variety $V(I)$ and use your description to show that $I \neq \sqrt{I}$.

6. Is the ring $k[x, y]/(y^2 - x^3)$ integrally closed in its field of fractions?

7. Suppose R is a commutative (unital) ring, M and N are R-modules and $f : M \to N$ is an R-module homomorphism. Show that f is surjective if and only if, for every prime ideal $\mathfrak{p} \subset R$, the induced map $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ of modules over $R_{\mathfrak{p}}$ is surjective.