MATH 507a **QUALIFYING EXAM** **January 2018.**

Answer all questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

When the problem asks you to compute something, you are expected to simplify the answer as much as possible.

1. Let $X_0, X_1, X_2, \ldots$ be a sequence of random variables such that $X_0 = 0$ and, for some $\gamma > 2$,

$$\sum_{k=0}^{\infty} k^{\gamma}\, \mathbb{E}(X_{k+1} - X_k)^2 < \infty$$

Prove that

$$\mathbb{P}\big(\sup_{k\geq 1} |X_k| < \infty\big) = 1.$$

2. (a) Let U and V be independent random variables. Assume that U has characteristic function $\varphi = \varphi(t)$ and assume that V has probability density function $f(x) = e^{-x},\ x \geq 0$. Express the characteristic function of the product UV in terms of φ.

(b) Let $\psi = \psi(t)$ be a characteristic function. Show that $\alpha = \alpha(t)$ defined by

$$\alpha(t) = \int_0^1 \psi(ts)\, ds$$

is also a characteristic function.

3. Let $X_1, X_2, \ldots$ be iid random variables with mean zero and variance one, and let $\tau_1, \tau_2, \ldots$ be a sequence of random variables, each taking only positive integer values, and such that, for a non-random number $c > 0$,

$$\lim_{n\to\infty} \frac{\tau_n}{n} = c$$

in probability. Define $S_n = \sum_{k=1}^{n} X_k$. Prove that

$$\lim_{n\to\infty} \frac{S_{\tau_n}}{\sqrt{\tau_n}} = \xi$$

in distribution, where ξ is a standard normal random variable. Note: there is no need to assume that the sequence $\{X_1, X_2, \ldots\}$ is independent of the sequence $\{\tau_1, \tau_2, \ldots\}$.