MATH 505a **QUALIFYING EXAM** **January 2018.**

Answer all questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

When the problem asks you to compute something, you are expected to simplify the answer as much as possible.

1. Let X and Y be independent standard normal random variables and define $V = \min(X, Y)$. Compute the probability density function of V^2. The final answer should be an elementary function.

2. Consider positions 1 to n arranged in a circle, so that 2 comes after 1, 3 comes after 2, $\ldots$, n comes after $n-1$, and 1 comes after n. Similarly, take 1 to n as values, with cyclic order, and consider all $n!$ ways to assign values to positions, bijectively, with all $n!$ possibilities equally likely. For $i = 1$ to n, let X_i be the indicator that position i and the one following are filled in with two consecutive values in increasing order, and define

$$S_n = \sum_{i=1}^{n} X_i, \quad T_n = \sum_{i=1}^{n} iX_i$$

For example, with $n = 6$ and the the circular arrangement 314562, we get $X_3 = 1$ since 45 are consecutive in increasing order, and similarly $X_4 = X_6 = 1$, so that $S_6 = 3, T_6 = 13$.

a) Compute the mean and the variance of S_n.

b) Compute the mean and the variance of T_n.

3. A box is filled with coins, each giving heads with some probability p. The value of p varies from one coin to another, and it is uniform in $[0, 1]$. A coin is selected at random; that one coin is tossed multiple times.

(a) Compute the probability that the first two tosses are both heads.

(b) Let X_n be the number of heads in the first n tosses. Compute $\mathbb{P}(X_n = k)$ for all $0 \le k \le n$.

(c) Let N be the number of tosses needed to get heads for the first time. Compute $\mathbb{P}(N = n)$ for all $n \ge 1$.

(d) Compute the expected value of N.

HINT: $\displaystyle\int_0^1 x^m(1-x)^\ell \, dx = \frac{m! \, \ell!}{(m+\ell+1)!}$ for nonnegative integers m, ℓ.