Numerical Analysis Screening Exam, Spring 2018

This exam has 4 problems. Please write down your names on all pages of your answer and mark clearly the order of pages. Start a new problem on a new page.

I. Direct Methods

Consider the matrix

$$A = \begin{bmatrix} R & \mid & v \\ u^T & \mid & 0 \end{bmatrix}$$

where R is an $n \times n$ invertible upper triangular matrix and u and v are vectors in $\mathbb{R}^n$.

(a) Consider the LU decomposition of A, $A = LU$, where L is unit lower triangular and U is upper triangular. Find the matrices L and U.

(b) Find necessary and sufficient conditions on R, u and v for A to be non-singular.

(c) Suppose A is non-singular. Use the LU-decomposition from (a) to formulate an economical algorithm for solving the system of equations $Ax = b$ and determine its computational cost.

II. Iterative Methods

Consider the system of equations $Ax = b$ where

$$A = D - E - F$$

is an $n \times n$ Hermitian positive definite matrix, D is diagonal, $-E$ is strictly lower triangular and $-F$ is strictly upper triangular. (In other words, both E and F have zeros on the diagonal.) To solve the system using the Gauss-Seidel method with successive over-relaxation (SOR) we write $A = M - N$ where

$$M = \frac{1}{\omega} D - E$$

$$\text{and} \qquad N = \left(\frac{1}{\omega} - 1\right) D + F$$

where ω is a real number. The iteration matrix is

$$T_\omega = M^{-1}N = (D - \omega E)^{-1} \left[(1 - \omega)D + \omega F\right].$$

(a) For any matrix B let B^* be its conjugate transpose. Show that if $|\omega - 1| < 1$ then $M^* + N$ is positive definite.

(b) Consider the vector norm $||\cdot||_A$ be defined by

$$||x||_A = \sqrt{x^* A x}.$$

and the corresponding matrix norm $||\cdot||_A$ it induces on $\mathbb{C}^{n \times n}$. Show that if $M^* + N$ is positive definite then $||M^{-1}N||_A < 1$.

(c) What can you deduce about the convergence of the SOR method in the case when $|\omega - 1| < 1$?

(d) Suppose the SOR method converges for all initial vectors x_0. Show that $|\omega - 1| < 1$. *(Hint: compute* $\det(T_\omega)$*.)*

III. Eigenvalues Problem

Suppose A is a real symmetric $n \times n$ matrix. Given two integers p and q between 1 and n, let

$$G = G(p,q,\theta) = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & c & \dots & s & & & \\ & & & \vdots & \ddots & \vdots & & & \\ & & & -s & \dots & c & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

be a Givens matrix. In other words, G is the $n \times n$ matrix defined by

$$\begin{array}{lll} G_{kk} = 1 \quad k \neq p,q & & G_{ij} = 0 \quad i \neq j \text{ and } (i,j) \neq (p,q) \text{ or } (q,p) \\ G_{pp} = c & & G_{pq} = s \\ G_{qq} = c & & G_{qp} = -s \end{array}$$

where $c = \cos\theta$ and $s = \sin\theta$.

(a) Let $A_1 = G^T AG$ and suppose θ is chosen so that $[A_1]_{pq} = 0$. Let $||\cdot||$ be the Frobenius norm given by

$$||M|| = \sum_{i,j} M_{ij}^2.$$

Let $A = D_0 + E_0$ and $A_1 = D_1 + E_1$ where D_0 and D_1 are diagonal matrices and E_0 and E_1 have zeros on their diagonals. Show that

$$||E_1||^2 = ||E_0||^2 - 2A_{pq}^2$$

(b) Use part (a) to describe an iterative method for finding the eigenvalues of A and prove your algorithm converges.

IV. Least Square Problem

Let $A \in \mathbb{R}^{m\times n}$ be a matrix and let $A = U\Sigma V^T$ be its singular value decomposition. We want to find k vectors $x_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^n$ such that

$$\left\| A - \sum_{i=1}^{k} x_i y_i^T \right\|_F^2$$

is minimal, where $\|A\|_F^2 := \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2$ is the Frobenius norm.

(a) Let u_i and v_j the i-th and j-th column of U and V respectively and σ_r the r-th singular value of A. Show that

$$x_i = \sqrt{\sigma_i} u_i, \qquad\qquad y_i = \sqrt{\sigma_i} v_i$$

is a solution of the optimization problem .

Hint: For any orthogonal matrix O and S we have $\|OAS\|_F = \|A\|_F$.

(b) Assume that $m = n$ is large and k small. How many values do you have to store for A and how many for its approximation $\sum_{i=1}^{k} x_i y_i^T$?

(c) Suppose that the matrix A represents a database of rankings of n movies by m movie viewers such that $a_{ij} \in \{0, \dots, 5\}$ is the star ranking of movie j by viewer i. Is it reasonable that A can be well approximated by some $\sum_{i=1}^{k} x_i y_i^T$ for some comparatively small k?

Hint: Discuss the case that there are k groups of users with similar interests and therefore similar movie rankings.