DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2017

Each problem is worth 10 points. There are seven problems.

1. Consider the 2^{nd} order ODE,

$$x'' + p(t)x' + a\,x = 0$$

where $a > 0$ and $\displaystyle\int_0^t p(s)ds \to \infty$ an $t \to \infty$. Suppose $\phi(t)$ and $\psi(t)$ form a fundamental set of solutions, i.e.

$$X(t) = \begin{pmatrix} \phi(t) & \psi(t) \\ \phi'(t) & \psi'(t) \end{pmatrix}$$

is non-singular. Prove $\det X(t) \to 0$ as $t \to \infty$.

2. Given the system of ODE's

$$x' = Ax + f(t, x),$$

where A is an $n \times n$ real matrix with spectrum in the open left half of the complex plane. The function f is real, continuous for small $|x|$ and $t \geq 0$ and

$$f(t, x) = o(|x|), \qquad \text{as} \quad |x| \to 0,$$

uniformly in t, i.e. given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\frac{|f(t, x)|}{|x|} \leq \varepsilon,$$

whenever $0 < |x| < \delta$, and $t \in [0, \infty)$. Prove that the zero solution is asymptotically stable.

3. Show that the planar system

$$\begin{aligned} x' &= x - y - x^3 - xy^2 \\ y' &= x + y - x^2y - y^3, \end{aligned}$$

has a periodic orbit Π in $\mathbb{R}^2$ other than the trivial one at $(0, 0)$. Is $(0, 0)$ necessarily contained in the region surrounded by Π? Justify your answer.

4. Consider the initial value problem

$$\begin{aligned} &u_t + uu_x = 2, \qquad x \in \mathbb{R},\ t > 0, \\ &u(x, 0) = x, \qquad x \in \mathbb{R}. \end{aligned}$$

   (a) Find the equation for the characteristics. Are there any shock forming with this initial condition?

   (b) Find an explicit formula for the solution of this initial value problem.

5. Let $U \subset \mathbb{R}^n$ be open and bounded and let $u \in C^2(U) \cap C(\bar{U})$ be harmonic in U. Prove that if U is connected and there exists $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$, then u is constant within U.

6. Let U be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with a smooth boundary and let u be a C^2-solution of the following initial boundary value problem

$$\begin{aligned} &u_t = \Delta u \quad \text{in } U \times (0, \infty), \\ &u = 0 \quad \text{on } \partial U \times (0, \infty), \\ &u = u_0 \quad \text{on } U \times \{t = 0\}. \end{aligned}$$

Here u_0 is smooth, bounded and integrable in U. Show that there exists a constant $\alpha > 0$ such that

$$\|u(\cdot, t)\|_{L^2(U)} \leq e^{-\alpha t} \|u_0\|_{L^2(U)}.$$

7. We define a weak solution of the one-dimensional wave equation $u_{tt} - u_{xx} = 0$ to be a function $u(x, t)$ such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t) \left(\phi_{tt}(x, t) - \phi_{xx}(x, t) \right) dx dt = 0$$

for every $\phi \in C_c^2(\mathbb{R}^2)$.

(a) Show that any C^2-solution of the wave equation is also a weak solution.

(b) Determine if $u(x, t) := H(x - t)$ is a weak solution of the wave equation or not. Here H is the Heaviside function: $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$.