1. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a vector of i.i.d. $N(\mu, \sigma^2)$ random variables, where both μ and σ are unknown.

   (a) Given $\alpha_1 \in (0,1)$, write down an exact $(1-\alpha_1)$ confidence interval for μ.

   (b) Given $\alpha_2 \in (0,1)$, write down an exact $(1-\alpha_2)$ confidence interval for σ^2.

   (c) Letting $\mathcal{I}_{\alpha_1}(\boldsymbol{X})$ and $\mathcal{J}_{\alpha_2}(\boldsymbol{X})$ denote the confidence intervals in parts 1a and 1b, respectively, for given $\alpha \in (0,1)$ show how to choose α_1, α_2 so that the overall coverage probability satisfies

   $$P_{\mu,\sigma^2}\left(\mu \in \mathcal{I}_{\alpha_1}(\boldsymbol{X}) \text{ and } \sigma^2 \in \mathcal{J}_{\alpha_2}(\boldsymbol{X})\right) \geq 1-\alpha \quad \text{for all} \quad \mu, \sigma^2.$$

   The inequality does not have to be sharp.

2. Let P_0 and P_1 be probability distributions on $\mathbb{R}$ with densities p_0 and p_1 with respect to Lebesgue measure, and let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables.

   (a) Let β denote the power of the most powerful test of size α, $0 < \alpha < 1$, for testing the null hypothesis $H_0 : X_1, \ldots, X_n \sim P_0$ against the alternative $H_a : X_1, \ldots, X_n \sim P_1$. Show that $\alpha < \beta$ unless $P_0 = P_1$.

   (b) Let P_0 be the uniform distribution on the interval $[0,1]$ and P_1 be the uniform distribution on $[1/3, 2/3]$. Find the Neyman-Pearson test of size α for testing H_0 against H_a (consider all possible values of $0 < \alpha < 1$).