1. Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, 1)$. Let $u \in \mathbb{R}$ be a given threshold value, and assume that we want to estimate the probability $p = p(\mu) = P_\mu(X_1 \leq u)$.

   (a) Find an unbiased estimator of p.

   (b) Letting $\overline{X} = n^{-1} \sum_{i=1}^n X_i$ denote the sample mean, show that the joint distribution of $\overline{X}$ and $X_1 - \overline{X}$ is bivariate normal and find the parameters of this distribution. Use your answer to demonstrate that $\overline{X}$ and $X_1 - \overline{X}$ are independent.

   (c) Use the estimator from part (1a), along with the Rao-Blackwell theorem and part (1b), to find the uniform minimal variance unbiased estimator (UMVUE) for p.

2. For all $k = 0, 1, \ldots,$ we have that

$$\int_{-\infty}^{\infty} x^k e^{-x^4/12} dx = 2^{\frac{k-3}{2}} 3^{\frac{k+1}{4}} \left((-1)^k + 1\right) \Gamma\left(\frac{k+1}{4}\right).$$

   (a) Determine c_1 such that

$$p(x) = c_1 e^{-x^4/12} \tag{1}$$

   is a density function.

   In the following we consider the location model

$$p(x; \theta) = p(x - \theta), \quad \theta \in (-\infty, \infty)$$

   where $p(x)$ is as in (1). Assume a sample $X_1, \ldots, X_n$ of independent random variables with distribution $p(x; \theta)$ has been observed.

   (b) Prove that the maximizer of the likehood function is unique and can be found by setting the derivative of the log likelihood to zero.

   (c) Determine the maximum likelihood estimate $\widehat{\theta}$ of θ based on the sample in a form as explicit as you can in terms of the sample moments

$$m_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k.$$

   (d) Determine the information $I_{\mathbf{X}}(\theta)$ for θ in the sample $X_1, \ldots, X_n$, and the non-trivial limiting distribution of $\widehat{\theta}$, when properly scaled and centered. You may assume regularity conditions hold without explicitly noting them.