Geometry and Topology Graduate Exam
Spring 2017

Problem 1. Let $\omega \in \Omega^2(M)$ be a differential form of degree 2 on a $2n$–dimensional manifold M. Suppose that ω is exact, namely that $\omega = d\alpha$ for some $\alpha \in \Omega^1(M)$. Show that $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is exact.

Problem 2. Consider the unit disk $B^2 = \{x \in \mathbb{R}^2; \|x\| \leq 1\}$ and the circle $S^1 = \{x \in \mathbb{R}^2; \|x\| = 1\}$. The two manifolds $U = S^1 \times B^2$ and $V = B^2 \times S^1$ have the same boundary $\partial U = \partial V = S^1 \times S^1$. Let X be the space obtained by gluing U and V along this common boundary; namely, X is the quotient of the disjoint union $U \sqcup V$ under the equivalence relation that identifies each point of ∂U to the point of ∂V that corresponds to the same point of $S^1 \times S^1$.

Compute the fundamental group $\pi_1(X; x_0)$.

Problem 3. Compute the homology groups $H_n(X; \mathbb{Z})$ of the topological space X of Problem 2.

Problem 4. For a unit vector $v \in S^{n-1} \subset \mathbb{R}^n$, let $\pi_v \colon \mathbb{R}^n \to v^\perp$ be the orthogonal projection to its orthogonal hyperplane $v^\perp \subset \mathbb{R}^n$. Let M be an m–dimensional submanifold of $\mathbb{R}^n$ with $m \leq \frac{n}{2} - 1$. Show that, for almost every $v \in S^{n-1}$, the restriction of π_v to M is injective.

Possible hint: Use a suitable map $f \colon M \times M - \Delta \to S^{n-1}$, where $\Delta = \{(x, x); x \in M\}$ is the diagonal of $M \times M$.

Problem 5. Let G be a topological group. Namely, G is simultaneously a group and a topological space, the multiplication map $G \times G \to G$ defined by $(g, h) \mapsto gh$ is continuous, and the inverse map $G \to G$ defined by $g \mapsto g^{-1}$ is continuous as well. Show that, if $e \in G$ is the identity element of G, the fundamental group $\pi_1(G; e)$ is abelian.

Problem 6. Let $f \colon M \to N$ be a differentiable map between two compact connected oriented manifolds M and N of the same dimension m. Show that, if the induced homomorphism $H_m(f) \colon H_m(M; \mathbb{Z}) \to H_m(N; \mathbb{Z})$ is nonzero, the subgroup $f_*\big(\pi_1(M; x_0)\big)$ has finite index in $\pi_1(N; f(x_0))$. Hint: consider a suitable covering of N.