REAL ANALYSIS GRADUATE EXAM

Spring 2017

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Assume that f is a positive absolutely continuous function on $[0, 1]$. Prove that $1/f$ is also absolutely continuous on $[0, 1]$.

2. Assume that E is Lebesgue measurable.
(i) Suppose $m(E) < \infty$, where m is the Lebesgue measure. Show that

$$f(x) = \int \chi_E(y)\chi_E(y - x)dm(y)$$

is continuous. (Here, χ_A denotes the characteristic function of a set $A \subseteq \mathbb{R}$).
(ii) Suppose $0 < m(E) \leq \infty$. Show that $S = E - E = \{x - y : x, y \in E\}$ contains an open interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$.

3. Assume that f is a continuous function on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 nx^{n-1} f(x)\, dx = f(1).$$

4. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let f, g be measurable real valued functions. Show that

$$\int |f - g|\, d\mu = \int_{-\infty}^{\infty} \int \left|\chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x))\right| d\mu(x)dt.$$