COMPLEX ANALYSIS GRADUATE EXAM

Spring 2017

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let γ be a circle with radius 1 and center 0 with positive direction of integration. Compute

$$\int_\gamma e^{1/z}\, dz.$$

2. Assume that f is a holomorphic function on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$f(z)^3 = \overline{f(z)}, \qquad \forall z \in D.$$

Prove that f is a constant.

3. Let $f(z) = \sum_{n=1}^{\infty} z^{n!}$.
(i) Show that $f(z)$ is holomorphic in the unit disk $D = \{z \in \mathbb{C} : |z < 1\}$.
(ii) Show that f does not have any holomorphic extension, that is, there exists no g holomorphic on some open set $U \supseteq D$ such that $U \neq D$ and $f = g|D$. (Hint: Consider $e^{i\theta}$ where θ is rational.)

4. Let f be an entire function such that

$$f(z + m + ni) = f(z), \qquad \forall z \in \mathbb{C}, \qquad \forall m, n \in \mathbb{Z}.$$

Prove that f is a constant function.