Algebra Exam January 2017

Show your work. Be as clear as possible. Do all problems.

1. Let R be a PID. Let M be an R-module.
   (a) Show that if M is finitely generated, then M is cyclic if and only if M/PM is for all prime ideals P of R.
   (b) Show that the previous statement is false if M is not finitely generated.

2. Prove that a power of the polynomial $(x+y)(x^2+y^4-2)$ belongs to the ideal (x^3+y^2, x^3+xy) in $\mathbb{C}[x,y]$.

3. Let G be a finite group with a cyclic Sylow 2-subgroup S.
   (a) Show that $N_G(S) = C_G(S)$.
   (b) Show that if $S \neq 1$, then G contains a normal subgroup of index 2 (hint: suppose that $n = [G : S]$, consider an appropriate homomorphism from G to S_n).
   (c) Show that G has a normal subgroup N of odd order such that $G = NS$.

4. Show that $\mathbb{Z}[\sqrt{5}]$ is not integrally closed in its quotient field.

5. Let $f(x) = x^{11} - 5 \in \mathbb{Q}[x]$.
   (a) Show that f is irreducible in $\mathbb{Q}[x]$.
   (b) Let K be the splitting field of f over $\mathbb{Q}$. What is the Galois group of $K/\mathbb{Q}$.
   (c) How many subfields L of K are there so such that $[K : L] = 11$.

6. Suppose that R is a finite ring with 1 such that every unit of R has order dividing 24. Classify all such R.