MATH 507a PROBABILITY GRADUATE EXAM
Spring 2017

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose that $X_1, X_2, X_3, \ldots$ are independent and that X_n has a uniform distribution on the interval $[a_n, b_n]$ for $n \geq 1$, where $a_n < b_n$. Find necessary and sufficient conditions on the sequences $\{a_n\}$ and $\{b_n\}$ so that $\sum_n X_n$ converges almost surely.

(2) Suppose that $X_1, X_2, X_3, \ldots$ are independent and identically distributed with distribution function $F(x)$ satisfying $x(1 - F(x)) \to c$ as $x \to \infty$ for some positive constant c. Define $M_n = \max(X_1, X_2, \ldots, X_n)$.

(i) Find the distribution function of M_n in terms of F.

(ii) Show that M_n/n converges in distribution as $n \to \infty$ and find the distribution function of the limit.

(3) Suppose that $X_1, X_2, \ldots$ are i.i.d. with $0 < E(X_1^2) < \infty$, and let $S_n = X_1 + X_2 + \cdots + X_n$.

(i) Show that $\dfrac{|X_n|}{\sqrt{n}} \to 0$ a.s.

(ii) Without using the Law of the Iterated Logarithm, show that $\limsup_{n\to\infty} \dfrac{|S_n|}{\sqrt{n}} = \infty$ a.s. HINT: How does this relate to the events $\left\{ \dfrac{|S_n|}{\sqrt{n}} > c \right\}$ with $c > 0$? Also, for fixed k, do the values $X_1, \ldots, X_k$ affect the lim sup?

(4) Suppose that $\{X_n\}$ is a family of random variables such that $EX_n^2 \leq C$ for some $C < \infty$.

(i) Show that the family $\{X_n\}$ is tight.

(ii) Suppose that $X_n \Rightarrow X$. Show that $EX^2 \leq C$ and that $EX_n \to EX$. (Here $\Rightarrow$ means convergence in distribution.)

(iii) Give an example where $X_n \Rightarrow X$ but $EX_n^2 \not\to EX^2$.