MATH 505a PROBABILITY GRADUATE EXAM
Spring 2017

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Three points are chosen independently and uniformly inside the unit square in the plane. Find the expected area of the smallest closed rectangle that has sides parallel to the coordinate axes and that contains the three points. HINT: Consider what happens with just one coordinate.

(2) Suppose (X, Y) has joint density of the form $f(x, y) = g(\sqrt{x^2 + y^2})$ for $(x, y) \in \mathbb{R}^2$, for some function g. Show that $Z = Y/X$ has the Cauchy density $h(t) = 1/(\pi(1 + t^2)),\ t \in \mathbb{R}$. HINT: Polar coordinates.

(3) Assume $\sqrt{3} < C < 2$. Consider a sequence $X_1, X_2, X_3, \ldots$ of random variables where X_1 is uniform on $[0, 1]$, and where the conditional distribution of X_{n+1} given X_n is uniform on $[0, CX_n]$.

(a) Find the conditional expectation of $(X_{n+1})^r$ given X_n, for $r \geq 1$.
(b) Show that X_n converges to 0 in mean but not in mean square.
(c) Show that X_n converges to 0 almost surely.

(4) Suppose that n boys and m girls are arranged in a row, and assume that all possible orderings of the $n + m$ children are equally likely.

(a) Find the probability that all n boys appear in a single block.
(b) Find the probability that no two boys are next to each other.
(c) Find the expected number of boys who have a girl next to them on both sides.