Numerical Analysis Screening Exam
Spring 2017

1. Let $A \in \mathbb{C}^{n\times n}$ and let $A_j \in \mathbb{C}^n$ $j = 1,2,\dots,n$ be the j^{th} column of A. Show that

$$|\det A| \le \prod_{j=1}^{n} \|A_j\|_1.$$

Hint: Let $D = diag(\|A_1\|_1, \|A_2\|_1, \dots, \|A_n\|_1)$, and consider $\det B$, where $B = AD^{-1}$.

2. a) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and let $A_j \in \mathbb{R}^n$ $j = 1,2,\dots,n$ be the j^{th} column of A. Use Gram-Schmidt to show that $A = QR$, where $Q$is orthogonal and $R$is upper triangular with $\|A_j\|_2^2 = \sum_{i=1}^{j} R_{i,j}^2 \qquad j = 1,2,\dots,n.$

   b) Given $A, Q, R \in \mathbb{R}^{n\times n}$as in part (a) above with $A = QR$, and given $b \in \mathbb{R}^n$, perform an operation count (of multiplications only) for solving the linear system $Ax = b$.

3. Consider the constrained least squares problem:

$$* \begin{cases} \min\limits_{x} ||Ax - b||_2 \\ subject\ to\ \ Cx = d \end{cases}$$

   where the $m \times n$ matrix A, the $p \times n$ matrix C, and the vectors $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$ are given.

   a) Show that the unconstrained least squares problem
   $$\min_{x} ||Ax - b||_2$$
   is a special case of the constrained least squares problem *.

   b) Show that the minimum norm problem
   $$\begin{cases} \min\limits_{x} ||x||_2 \\ subject\ to\ \ Cx = d \end{cases}$$
   is a special case of the constrained least squares problem *.

   c) By writing $x = x_0 + Nz$, show that solving the constrained least squares problem * is equivalent to solving an unconstrained least squares problem
   $$** \min_{z} ||\tilde{A}z - \tilde{b}||_2 \ .$$
   What are the matrices N and $\tilde{A}$ and vectors x_0 and $\tilde{b}$?

   d) Use part c) to solve the constrained least squares problem * where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 4 & 3 \\ 0 & 0 & 2 \\ 1 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & 16 & 12 \\ -2 & -8 & -6 \\ 1 & 4 & 3 \\ -1 & -4 & -3 \end{bmatrix}, \qquad d = \begin{bmatrix} 12 \\ -6 \\ 3 \\ -3 \end{bmatrix}$$

4. Consider a stationary iteration method for solving a system of linear equations $Ax = b$ given by

$$y^k = x^k + \omega_0(b - Ax^k), \qquad x^{k+1} = y^k + \omega_1(b - Ay^k).$$

a) Show that the matrix B defined by $x^{k+1} = Bx^k + c$ has the form $B = \mu p(A)$ where $p(\lambda)$ is a second order polynomial in λ with leading coefficient equal to 1.

b) Show that the scaled Chebyshev polynomial $T_2(\lambda) = \lambda^2 - 1/2$ has the property that

$$\frac{1}{2} = \max_{-1 \le \lambda \le 1} |T_2(\lambda)| \le \max_{-1 \le \lambda \le 1} |q(\lambda)|$$

for all second order polynomial q with leading coefficient 1.

c) If we know that matrix A is Hermitian with eigenvalues in $(-1, 1)$, find coefficients ω_0 and ω_1 such that the proposed iterative scheme converges for any initial vector x^0.

d) What could you do if the eigenvalues of the matrix A is in $(\alpha, \beta)$to make the scheme convergent?