REAL ANALYSIS GRADUATE EXAM

Spring 2016

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let

$$f(y) = \sum_n \frac{x}{x^2 + yn^2}.$$

Show that $g(y) = \lim_{x\to\infty} f(x,y)$ exists for all $y > 0$. Find $g(y)$.

2. Let $A \subseteq \mathbb{R}$ be Lebesgue measurable. Show that $n(\chi_A * \chi_{[0,\frac{1}{n}]}) \to \chi_A$ pointwise a.e. as $n \to \infty$. (Recall that $(f * g)(x) = \int f(x-y)g(y)\,dy$ for $x \in \mathbb{R}$.)

3. a) Prove that if a sequence of integrable functions f_n on $[0,1]$ satisfies $\int_0^1 |f_n(x)|\,dx \leq 1/n^2$ for $n \in \mathbb{N}$, then $f_n \to 0$ a.e. on $[0,1]$ as $n \to \infty$.
b) Show that the above fact is not true if $1/n^2$ is replaced by $1/\sqrt{n}$.

4. Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Also, let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. Define

$$g_n(x) = \frac{1}{2^n} f(x - r_n), \qquad x \in \mathbb{R}$$

and let

$$g(x) = \sum_{n=1}^{\infty} g_n(x), \qquad x \in \mathbb{R}$$

a) Prove that g is integrable on $\mathbb{R}$.
b) Prove that g is discontinuous at every point in $\mathbb{R}$.