MATH 507a GRADUATE EXAM

Spring 2016

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let X and Y be random variables defined on the same probability space, and suppose that $X+Y$ has the same distribution as X, that is, for all t,

$$P(X+Y \le t) = P(X \le t). \tag{1}$$

(a) Does it follow that $P(Y=0)=1$?

(b) If we assume $Y>0$, does (1) then imply that $P(Y=0)=1$?

(c) If (instead of $Y>0$) we assume X and Y are independent, does (1) then imply that $P(Y=0)=1$?

In each case, either give a proof or construct a counterexample.

HINTS (IN NO PARTICULAR ORDER):

(i) Recall that X and $X+Y$ have the same distribution if and only if $E(f(X)) = E(f(X+Y))$ for all bounded continuous functions f on $\mathbb{R}$.

(ii) You may use the following theorem: If a random variable Z has characteristic function φ_Z satisfying $\varphi_Z(t)=1$ for all $t \in (-\delta,\delta)$ for some $\delta>0$, then $P(Z=0)=1$.

(iii) For one of the parts it may be useful to consider symmetric X.

(2) Let n points be i.i.d., uniformly distributed on the unit circle. Let Δ_n be the smallest distance between any two of these points. Show that $n^\theta \Delta_n \to 0$ in probability as $n \to \infty$, for all $0<\theta<2$. HINT: Divide the circle into small arcs and find the probability that at least one arc contains 2 or more points.

(3) Let $X_1, X_2, \ldots$ be iid with $P(X_1>0)=1$. Define $S_n = X_1 + \cdots + X_n$.

(a) Suppose $E(X_1)<\infty$. Show that

$$(*) \qquad \lim_{n\to\infty} \frac{X_n}{S_n} = 0 \quad \text{a.s.}$$

(b) Construct an example in which $E(X_1)=\infty$ and $(*)$ in part (a) is false.

HINT for (b): Let X_1 take values $a_1 < a_2 < \ldots$ and let $p_n = P(X_1 = a_n)$. For each k, consider the least n for which $X_n \ge a_k$. What is the maximum possible value of S_{n-1} for this n? Use this to choose the a_k's and p_k's so that, for this sequence of n's, $(*)$ will fail.

If you can't find a specific choice of a_k's and p_k's, at least try to describe qualitatively how these sequences need to behave, for example should they grow, or approach 0, rapidly or slowly, and why.