1. Suppose

$$A = \begin{pmatrix} D_1 & F_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ E_2 & D_2 & F_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & E_3 & D_3 & F_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & E_{n-1} & D_{n-1} & F_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & E_n & D_n \end{pmatrix}$$

is a real block tridiagonal matrix where the blocks are all size $q \times q$ and the diagonal blocks D_i are all invertible, $1 \leq i \leq n$. Suppose, moreover, that A^t is block diagonally dominant, in other words

$$||D_i^{-1}||_1 \left(||F_{i-1}||_1 + ||E_{i+1}||_1\right) < 1$$

for $1 \leq i \leq n$ where $F_0 = E_{n+1} = 0$.

(a) Show A is invertible.

(b) Show A has a block LU decomposition of the form

$$A = \begin{pmatrix} I & 0 & 0 & \dots & 0 & 0 \\ L_2 & I & 0 & \dots & 0 & 0 \\ 0 & L_3 & I & \dots & 0 & 0 \\ \dots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & L_n & I \end{pmatrix} \begin{pmatrix} U_1 & F_1 & 0 & \dots & 0 & 0 \\ 0 & U_2 & F_2 & \dots & 0 & 0 \\ 0 & 0 & U_3 & \dots & 0 & 0 \\ \dots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & U_{n-1} & F_{n-1} \\ 0 & 0 & 0 & \dots & 0 & U_n \end{pmatrix}$$

where

- $||L_i||_1 \leq 1$, $2 \leq i \leq n-1$ and
- each matrix U_i is invertible with $||U_i||_1 \leq ||A||_1$, $i \leq i \leq n$.

Hint: Recall, if a square matrix M has $||M||_1 < 1$ then $I - M$ is invertible and

$$(I - M)^{-1} = I + M + M^2 + M^3 + \dots$$

(c) Show how you can find this block LU decomposition numerically and how you can use it to solve the system of equations $Ax = b$ (for a given vector b). Explain the significance of the bounds in (b) and why this approach might be preferable to employing Gaussian elimination with pivoting on the whole of A.

2. Consider the following 2×3 matrix A

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

(a) Find a two dimensional subspace S^* such that

$$\min_{x \in S^*, \|x\|_2 = 1} \|Ax\|_2 = \max_{\dim S = 2} \min_{x \in S, \|x\|_2 = 1} \|Ax\|_2.$$

Justify your answer.

(b) Find a rank one 2×3 matrix B such that $\|A - B\|_2$ is minimized and justify your answer.

3. Let X be a linear vector space over C and let P be the $n \times n$ matrix defined by the linear transformation on X^n given by

$$P\vec{x} = P[x_1, x_2, \dots, x_n]^T = [x_2, x_3, \dots, x_n, x_1]^T.$$

(a) What are the matrices P, P^0, P^2, P^{n-1} and P^n ? (Hint: Although you could do this with matrix multiplication, it's easier to base your answer on the underlying transformation.)

Let F be the $n \times n$ matrix given by $[F]_{j,k} = \frac{1}{\sqrt{n}}\overline{\omega}^{(j-1)(k-1)}, j, k = 1,2, \dots, n$, where $\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

(b) Show that for $k = 1,2, \dots, n$, $P\vec{F}_k = \overline{\omega}^{(k-1)}\vec{F}_k$, where $\vec{F}_k$ is the k^{th} column of the matrix F.

(c) Show that the matrix F is unitary.

Let $a = \{a_i\}_{i=1}^n \subseteq C$, set $p_a(z) = \sum_{i=1}^n a_i z^{i-1}$ and let the $n \times n$ matrix A_a be given by

$$A_a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_n & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & a_2 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_n & a_1 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_1 \end{bmatrix}.$$

(d) Show that A_a is diagonalizable with eigenvalue/eigenvector pairs given by $\{p_a(\overline{\omega}^{(k-1)}), \quad \vec{F}_k\}$, $k = 1,2, \dots, n$. (Hint: Parts (a), (b) and (c).)

4. Let $\{z_k^0\}_{k=1}^n$ be n points in the complex plane and consider the following iteration:

$$z_k^{m+1} \text{ is equal to the average of} \begin{cases} z_k^m \text{ and } z_{k+1}^m & k = 1,2,\dots,n-1 \\ z_n^m \text{ and } z_1^m & k = n \end{cases}.$$

(a) Let $Z^m = [z_1^m, z_2^m, \dots, z_n^m]^T$ and rewrite the transformation from Z^m to Z^{m+1} given above in the form of a matrix iteration.

(b) Show that $\lim_{m\to\infty} z_k^m = \hat{z}$, $k = 1,2,\dots,n$, where $\hat{z} = \frac{1}{n}\sum_{j=1}^n z_j^0$. (Hint: The **RESULT** of Problem 3(d) might be of some help to you here. Note that you may use the result of problem 3(d) even if you were not able to prove it yourself.)

(c) What happens if, in parts (a) and (b) , the phrase "the average" in the definition of the iteration is replaced with "an arbitrary convex combination" ; that is:

$$z_k^{m+1} = \begin{cases} \alpha\, z_k^m + (1-\alpha)\, z_{k+1}^m & k = 1,2,\dots,n-1 \\ \alpha\, z_n^m + (1-\alpha)\, z_1^m & k = n \end{cases} \quad \text{for some } \alpha \in (0,1)?$$