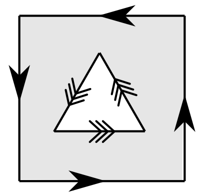
**Geometry and Topology Graduate Exam**
Spring 2016

**Problem 1.** Let $Y$ be the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let $X$ be the quotient space of $Y$ by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of $X$.



**Problem 2.** Let $X$ be a path connected space with $\pi_1(X; x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that $\widetilde{X}$ has either 2 or 6 connected components.

**Problem 3.** Compute the homology groups $H_k(S^1 \times S^n; \mathbb{Z})$ of the product of the circle $S^1$ and the sphere $S^n$, with $n \geq 1$.

**Problem 4.** Let $M$ be a compact oriented manifold of dimension $n$, and consider a differentiable map $f \colon M \to \mathbb{R}^n$ whose image $f(M)$ has non-empty interior in $\mathbb{R}^n$.

(a) Show there there is at least one point $x \in M$ where $f$ is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of $x$ such that restriction $f_{|U} \colon U \to f(U)$ is a diffeomorphism.

(b) Show that there exists at least two points $x$, $y \in M$ such that $f$ is a local diffeomorphism at $x$ and $y$, $f$ is orientation-preserving at $x$, and $f$ is orientation-reversing at $y$. Possible hint: What is the degree of $f$?

**Problem 5.** Consider the real projective space $\mathbb{RP}^n$, quotient of the sphere $S^n$ by the equivalence relation that identifies each $x \in S^n$ to $-x$. Is there a degree $n$ differential form such $\omega \in \Omega^n(\mathbb{RP}^n)$ such that $\omega(y) \neq 0$ at every $y \in \mathbb{RP}^n$? (The answer may depend on $n$.)

**Problem 6.** Let $S^n$ denote the $n$–dimensional sphere, and remember that for $n \geq 1$ its de Rham cohomology groups are

$$H^k(S^n) \cong \begin{cases} 0 & \text{if } k \neq 0, n \\ \mathbb{R} & \text{if } k = 0, n. \end{cases}$$

Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree $n$ on $S^n$ such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map $f$.

(a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

(b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of $\beta$ and $\alpha$.