DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2016

Each problem is worth 10 points. There are seven problems.

1. Find the stationary points and analyze their behavior as r varies from -1 to 1. Also determine the stability of each fixed point for each r.

$$\begin{aligned} x' &= rx - x^3 - y^2 - axy^3 \\ y' &= -y - x^2y^3 \end{aligned}$$

If there is a bifurcation then draw a bifurcation diagram.

2. Given the ODE

$$\frac{d^2x}{dt^2} + x + x^3 = 0,$$

choose the correct answer and prove it is correct:

- There exists a solution that $\to 0$ as $t \to \infty$.
- All non identically zero solutions are bounded and bounded away from zero,
- There exists a solution that $\to \infty$ as $t \to \infty$.

3. Show that the planar system

$$\begin{aligned} x' &= xy^2 + y^4 + x^3 \\ y' &= x^2y + 2y^5, \end{aligned}$$

can have no periodic orbits in $\mathbb{R}^2$ other than the trivial one at $(0,0)$.

4. Let

$$u(x) = \log\log\left(1 + \frac{1}{|x|}\right).$$

Show that u is unbounded and that it is an element of the Sobolev space $H^1(B(0,1))$, where $B(0,1)$ is the unit ball in $\mathbb{R}^2$.

5. Let Ω be a bounded open set of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function such that its derivative is bounded ($|f'| \leq K$) and $f(0) = 0$. Assume that u is a C^2 solution of

$$\begin{aligned} \partial_t u - \Delta u = f(u) \qquad & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 \qquad & \text{on } \partial\Omega \times (0,\infty). \end{aligned}$$

(i) Show that if $u(x,0) \geq 0$ for all $x \in \Omega$ then $u(x,t) \geq 0$ for all $x \in \Omega$ and all $t > 0$.

(ii) Show that if $u(x,0) \leq M$ for all $x \in \Omega$ then $u(x,t) \leq Me^{Kt}$ for all $x \in \Omega$ and all $t > 0$.

6. Consider the initial value problem

$$u_t + uu_x + u = 0, \qquad -\infty < x < \infty,\ t > 0 \tag{B}$$

subject to the initial condition $u(x,0) = a\sin x$.

(i) Find the characteristic curves associated with the equation (B) in an explicit form.

(ii) Show that if $a > 1$, then there exists a time $t = t(a)$ such that there exists no smooth solution (C^1in both space and time) of the equation (B) for $t > t(a)$. Find the maximal time of smoothness, $t(a)$.

7. Find the solution in the first quadrant $x > 0$ and $t > 0$ of the Wave equation with boundary condition at $x = 0$,

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \quad t > 0, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), \\ u_t(t,0) = au_x(t,0), \quad a \neq -c, \end{aligned}$$

where $f(x)$ and $g(x)$ are C^2 functions which vanish near $x = 0$. Show that no solution exists in general if $a = -c$.