

# COMPLEX ANALYSIS GRADUATE EXAM

Spring 2016

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let  $a \in \mathbb{C}$  be such that  $0 < |a| < 1$ , and set

$$f(z) = \frac{1 - z^2}{z^2 - (a + \frac{1}{a})z + 1}.$$

Find the Laurent expansion of  $f$  in a neighborhood of the unit circle  $|z| = 1$ .

2. Let  $a \in \mathbb{C}$  be such that  $0 < |a| < 1$  and let  $n \in \mathbb{N}$ . Show that  $e^z(z - 1)^n = a$  has exactly  $n$  simple roots in the half-plane  $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ .

3. Evaluate

$$\int_0^\infty \frac{\log^2 x}{1 + x^2} dx.$$

4. Denote  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Assume that  $f: \mathbb{D} \rightarrow \mathbb{D}$  is analytic. Show that if  $z_1 \neq z_2$  are fixed points of  $f$  in  $\mathbb{D}$ , then  $f$  is the identity map.