Spring 2015 Math 541b Exam

1. Let $X_1, X_2, \dots, X_n$ be independent Cauchy random variable with density

$$f(x|\theta) = \frac{1}{\pi(1 + (x-\theta)^2)},$$

and let $\widetilde{X_n} =$ median of $\{X_1, X_2, \dots, X_n\}$.

   (a) Prove that $\sqrt{n}(\widetilde{X_n} - \theta)$ is asymptotically normal with mean 0 and variance $\pi^2/4$ by showing that as n tends to infinity,

$$P(\sqrt{n}(\widetilde{X_n} - \theta) \le a) \longrightarrow P(Z \ge -2a/\pi)$$

   where Z is a standard normal random variable. *Hint:* If we define Bernoulli random variables $Y_i = 1_{\{X_i \le \theta + a/\sqrt{n}\}}$, the event $\{\widetilde{X_n} \le \theta + a/\sqrt{n}\}$ is equivalent to $\{\sum_i Y_i \ge (n+1)/2\}$ when n is odd. Applying the CLT might also be needed.

   (b) Using the result from part (a), find an *approximate* α-level large sample test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \ne \theta_0$.

2. We observe independent Bernoulli variables $X_1, X_2, \dots, X_n$, which depend on unobservable variables $Z_1, \dots, Z_n$ which, given $\theta_1, \dots, \theta_n$, are distributed independently as $N(\theta_i, 1)$, where

$$X_i = \begin{cases} 0 & \text{if } Z_i \le u \\ 1 & \text{if } Z_i > u. \end{cases}$$

The values $\theta_1, \theta_2, \dots, \theta_n$ are distributed independently as $N(\xi, \sigma^2)$. Assuming that u and σ^2 are known, we are interested in the maximum likelihood estimate of ξ.

   (a) Show that for any for given values of ξ and σ^2, and all $i = 1, \dots, n$, the random variable Z_i is normally distributed with mean ξ and variance $\sigma^2 + 1$.

   (b) Write down the likelihood function for the complete data $Z_1, \dots, Z_n$ when these values are observed.

(c) Now assume that only $X_1, \ldots, X_n$ are observed, and show that the EM sequence for the estimation of the unknown ξ is given by

$$\xi^{(t+1)} = \frac{1}{n} \sum_{i=1}^{n} E(Z_i | X_i, \xi^{(t)}, \sigma^2).$$

Start by computing the expected log likelihood of the complete data.

(d) Show that

$$E(Z_i | X_i, \xi^{(t)}, \sigma^2) = \xi^{(t)} + \sqrt{\sigma^2 + 1} \cdot H_i \left(\frac{u - \xi^{(t)}}{\sqrt{\sigma^2 + 1}} \right)$$

where

$$H_i(t) = \begin{cases} \frac{\phi(t)}{1 - \Phi(t)} & \text{if } X_i = 1 \\ -\frac{\phi(t)}{\Phi(t)} & \text{if } X_i = 0, \end{cases}$$

and $\Phi(t)$ and $\phi(t)$ are cumulative distribution and density function of a standard normal variable, respectively.