Spring 2015 Math 541a Exam

1. For $n \geq 2$ let $X_1, \cdots, X_n$ be independent samples from P_θ, the uniform distribution $U(\theta, \theta+1), \theta \in \mathbb{R}$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics of the sample.

   (a) Show that $(X_{(1)}, X_{(n)})$ is a sufficient statistic for θ.

   (b) Is $(X_{(1)}, X_{(n)})$ complete? Prove your claim.

   (c) Find a_n and $b(\theta)$ such that $a_n(b(\theta) - X_{(n)}) \to Z$ in distribution, where Z has an exponential distribution with density $f(x) = e^{-x}, x > 0$.

   (d) What is the maximum likelihood estimate of θ given the sample?

2. Let X and Y be independent random variables with $X \sim \text{exponential}(\lambda)$ and $Y \sim \text{exponential}(\mu)$, where the exponential$(\nu)$ density is given by

$$f(x;\nu) = \frac{1}{\nu} \exp\left(-\frac{x}{\nu}\right).$$

   Let

$$Z = \min\{X, Y\},$$

   and

$$W = 1 \text{ if } Z = X, \text{ and } W = 0 \text{ otherwise.}$$

   (a) Find the joint distribution of Z and W.

   (b) Prove that Z and W are independent.

   (c) Suppose that (X, Y) are not observable. Instead, with $n \geq 2$, we observe $(Z_1, W_1), \cdots, (Z_n, W_n)$, independent samples distributed as (Z, W). Write down the likelihood function in terms of the sample averages $(\overline{Z}, \overline{W})$, and find the maximum likelihood estimate $(\widehat{\mu}_n, \widehat{\lambda}_n)$ of (μ, λ).

   (d) Determining whether $1/\hat{\lambda}_n$ is unbiased for $1/\lambda$, and if not, construct an estimator that is.