MATH 505a QUALIFYING EXAM Monday, February 9, 2015. One hour and 50 minutes, starting at 5pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let X_n, $n \geq 1$, be independent random variables such that each X_n has Poisson distribution with mean λ_n. Prove that if $\sum_{n\geq 1} \lambda_n = +\infty$, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n \lambda_k} = 1$$

in probability.

2. A deck of cards is shuffled thoroughly. Someone goes through all 52 cards, scoring 1 each time 2 cards of the same value are consecutive. For example 9H,8H,7D,6C,7S,7H,7C, scores 2, once due to 7 of spades next to 7 of hearts, and once more 7 of hearts next to 7 of clubs. Write X for the total score.

a) Compute $\mathbb{E}X$.

b) Compute $\mathrm{Var}X$.

c) Compute $\mathbb{P}(X = 39)$.

d) In the line below, circle the number that you think is the closest to the value $\mathbb{P}(X = 0)$ and briefly explain your choice.

$$\frac{1}{1000}, \quad \frac{1}{500}, \quad \frac{1}{100}, \quad \frac{1}{50}, \quad \frac{1}{20}, \quad \frac{1}{10}, \quad \frac{1}{5}, \quad \frac{1}{2}.$$

3. Let $S_0, S_1, S_2, \ldots$ be a simple symmetric random walk, i.e. $\mathbb{P}(S_i - S_{i-1} = 1) = \mathbb{P}(S_i - S_{i-1} = -1) = 1/2$, with independent increments. Let $T = \min\{n > 0 : S_n = 0\}$ be the hitting time to zero. Write $\mathbb{P}_a$ for probabilities for the walk starting with $S_0 = a$.

a) What does the reflection principle say about $\mathbb{P}_a(S_n = i, T \leq n)$, for $a > 0$, and $i, n \geq 0$?

b) What does the reflection principle say about $\mathbb{P}_a(S_n \geq i, T > n)$, for $a > 0$, and $i, n \geq 0$? [Hint: telescoping series]

c) For fixed $a > 0$, give asymptotics for $\mathbb{P}_a(T > n)$ as $n \to \infty$. [HINT: Stirling's formula is that $n! \sim \sqrt{2\pi n}\,(n/e)^n$.]

d) Simplify, for fixed $a > 0$,

$$\lim_{n\to\infty} \frac{\mathbb{P}_{a+1}(T > n)}{\mathbb{P}_a(T > n)}.$$