Exam in Numerical Analysis Spring 2015

Instructions The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. **Linear systems**

   Consider the symmetric positive definite (spd) $n \times n$ matrix A and its LU-decomposition $A = LU$ with $l_{ii} = 1$, $i = 1,2,\ldots,n$.

a. Show that $u_{11} = a_{11}$, and $u_{kk} = \frac{\det(A_k)}{\det(A_{k-1})}$, $k = 2,\ldots,n$, where for each $k = 1,2,\ldots,n$, A_k is the $k \times k$matrix formed by taking the intersection of the first $k$rows and $k$columns of A.

b. Show that A can be written as $A = R^T R$ with R upper triangular with positive diagonal entries. (Hint: Let $D = \text{diag}(u_{11}, u_{22}, \ldots, u_{nn})$ and consider the identity $A = LDD^{-1}U$.)

c. Show how the decomposition found in part (b) suggests a scheme for solving the system $Ax = b$ with A spd, that like Choleski's method requires *only* U, but that unlike Choleski's method, does *not* require the computation of square roots.

2. **Least squares**

   Let $A \in R^{m \times n}$, $m \geq n$, be given.

   (a) Show that the x-component of any solution of linear system

$$\begin{pmatrix} I_m & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} , \qquad (1)$$

   is a solution of the minimization problem

$$\min_x ||b - Ax||_2. \qquad (2)$$

   (b) Show that the solution of the linear system (1) is unique if and only if the solution of the least squares problem (2) is unique.

3. **Iterative Methods**

   Let A be an $n \times n$ nonsingular matrix and consider the matrix iteration

$$X_{k+1} = X_k + X_k(I - AX_k),$$

   with X_0 given. Find and justify necessary and sufficient conditions on A and $X_0$for this iteration to converge to A^{-1}.

4. Computation of Eigenvalues and Eigenvectors

Consider the matrices A and T given by

$$A = \begin{bmatrix} 3 & \alpha & \beta \\ -1 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix},$$

where $|\alpha|, |\beta| \leq 1$.

(a) Use the similarity transform $T^{-1}AT$ to show that the matrix A has at least two distinct eigenvalues. (Hint: Gershgorin's Theorem)

(b) In the case $\alpha = \beta = 1$verify that $x = [1 \quad 1 \quad 1]^T$is an eigenvector of A and find a unitary matrix U such that the matrix $U^T AU$ has the form

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

(c) Explain, without actually calculating, how one could find a Schur decomposition of A.