DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2015

1. Consider the initial value problem for the one dimensional wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \ u_t = h & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

   Prove that if $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$, but not in general smoother.

2. Consider the Euler equations for the flow of a constant density, inviscid fluid in three dimensions:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P; \qquad \nabla \cdot u = 0,$$

   where the vector $u(x,t)$ is the fluid velocity and the scalar $P(x,t)$ is the pressure.

   Let $(\overline{u}(x), \overline{P}(x))$ be a stationary solution of the system (steady flow). Prove that the quantity

$$H = \overline{P} + \frac{1}{2}\overline{u}^2$$

   is constant along the curves $dx/dt = \overline{u}(x)$.

3. Let $f(r) = r^5 - 4r^3$. Show that if u is a solution of the problem

$$\begin{cases} \Delta u = f(u) & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

   where U is a bounded open subset of $\mathbb{R}^n$, then necessarily $|u(x)| < 2$ throughout U.

4. Show that all solutions of

$$u'' + 4u + 3u^5 = 0$$

   are bounded. If $u(0) = 1$ and $u'(0) = 2$ give a sharp bound on $u'(t)$, i.e. the least upper bound for $|u'(t)|$.

5. Consider the system of ODE's,

$$\begin{aligned} x_1' &= -x_1 - x_1 x_2^2, \\ x_2' &= \mu x_2 - 2x_3 - x_2(x_2^2 + x_3^2), \\ x_3' &= 2x_2 + \mu x_3 - x_3(x_2^2 + x_3^2). \end{aligned}$$

   Ler S be the set of initial conditions close to the origin whose solutions approach the origin as $t \to \infty$. (1) For $\mu = 1$ determine the dimension of S. (2) Describe the qualitative change that takes place when μ is varied continuously from -1 to $+1$.

6. Consider the periodic system with period T,

$$x' = A(t)x, \qquad A(t + T) = A(t).$$

   Show there is a solution $\phi(t)$ and a constant λ such that $\phi(T) = \lambda\phi(0)$.