COMPLEX ANALYSIS GRADUATE EXAM

Spring 2015

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx,$$

being careful to justify your answer.

2. Let

$$f(z) = \sum_{n=0}^\infty z^{n!}.$$

(i) Show that f is analytic in the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.
(ii) Show that f can not be analytically continued to any open set properly containing D. (Hint: First consider $z = r^{2\pi i p/q}$ where p and q are integers.)

3. Let A be an open subset of $\mathbb{C}$, and suppose $u(x, y)$ is a twice continuously differentiable harmonic function on A.
(i) Show that if A is simply connected, then there exists an analytic function f on A such that $u = \mathbb{R}\mathrm{e}\, f$. (Hint: First find g so that $\partial u/\partial x = \mathbb{R}\mathrm{e}\, g$.)
(ii) Find f explicitly when $A = \mathbb{C}$ and $u(x, y) = e^x \cos y + xy$.
(iii) Give an example in which A is not simply connected and f as in (i) does not exist.

4. Determine whether it is possible for a function f to be analytic in a neighborhood of 0 and take the values $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \ldots$ at the points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8} \ldots$.