Algebra Exam February 2015

Show your work. Be as clear as possible. Do all problems.

1. Use Sylow's theorems and other results to describe, up to isomorphism, the possible structures of a group of order 1005.

2. Let R be a commutative ring with 1. Let M, N and V be R-modules.

   (a) Show if that M and N are projective, then so is $M \otimes_R N$.

   (b) Let $\mathrm{Tr}(V) := \{\sum_i \phi_i(v_i) | \phi \in \mathrm{Hom}_R(V, R), v_i \in V\} \subset R$. If $1 \in \mathrm{Tr}(V)$, show that up to isomorphism R is a direct summand of V^k for some k.

3. Let F be a field and M a maximal ideal of $F[x_1, \ldots, x_n]$. Let K be an algebraic closure of F. Show that M is contained in at least 1 and in only finitely many maximal ideals of $K[x_1, \ldots, x_n]$.

4. Let F be a finite field.

   (a) Show that there are irreducible polynomials over F of every positive degree.

   (b) Show that $x^4 + 1$ is irreducible over $\mathrm{Q}[x]$ but is reducible over $\mathbb{F}_p[x]$ for every prime p (hint: show there is a root in $\mathbb{F}_{p^2}$).

5. Let F be a field and M a finitely generated $F[x]$-module. Show that M is artiniian if and only if $\dim_F M$ is finite.

6. Let R be a right Artinian ring with with a faithful irreducible right R-module. If $x, y \in R$, set $[x, y] := xy - yx$. Show that if $[[x, y], z] = 0$ for all $x, y, z \in R$, then R has no nilpotent elements.