Spring 2006 Math 541b Exam

1. Let $X_1, \cdots, X_n$ be i.i.d. samples from a Weibull distribution with density $f(x, \lambda) = \lambda\, c x^{c-1} e^{-\lambda x^c}$, where $x > 0$, and c is a known positive constant and $\lambda > 0$ is the scale parameter of interest. Let $\mu = 1/\lambda$.

   (a) Show that $\sum_{i=1}^n X_i^c$ is an optimal test statistic or testing H: $\mu = \mu_0$ versus K: $\mu = \mu_1 > \mu_0$. That is, the most powerful test takes the form:
   $$\begin{cases} \text{reject } H & \text{if } \sum_{i=1}^n X_i^c > \text{critical value} \\ \text{accept } H & \text{if } \sum_{i=1}^n X_i^c \leq \text{critical value}\,. \end{cases}$$

   (b) Show that λX_i^c follows the standard exponential distribution Exp(1).

   (c) Find the critical value for the size α most powerful test.

   (d) Show that the power of the most powerful test of size α is given by
   $$\beta(\mu_1) = 1 - G_n(\frac{\mu_0}{\mu_1} g_n(1-\alpha))\,.$$
   where G_n is the distribution function of $\Gamma(n, 1)$, $g_n(1-\alpha)$ is the $(1-\alpha)$th quantile of $\Gamma(n, 1)$, and prove that $\beta(\mu)$ is increasing in μ.

   (e) Show that the most powerful test of size α for the simple hypotheses in (a) is uniformly most powerful, at size α, for testing the composite hypotheses H: $\mu \leq \mu_0$ versus K: $\mu > \mu_0$.

   (f) When n is large, please use normal approximation to find the critical value and power.

2. Let $X_i, B_i, i = 1, \ldots, n$ be independent Bernoulli variables where X_i has unknown success probability $p \in (0, 1)$, and B_i has success probability 1/3. Suppose we observes
   $$Y_i = B_i X_i + (1 - B_i)(1 - X_i), \quad i = 1, \ldots, n$$
   that is, we see the original X_i with probability 1/3, and $1 - X_i$ with probability 2/3.

   (a) Write the log likelihood in term so the sum $S_n = \sum_{i=1}^n Y_i$, and the equation one would solve for finding the maximum likelihood estimator.

(b) Introduce appropriate missing data for the implementation of the EM algorithm and write out the full likelihood, and the maximum likelihood estimator using this data.

(c) Detail the steps of the EM algorithm.