MATH 507 **QUALIFYING EXAM** **January 31, 2006**

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1. Suppose that $X_1, X_2, \ldots$ are independent, with $\mathbb{P}(X_n = 1) = p_n = 1 - \mathbb{P}(X_n = 0)$.

a) Find and prove a necessary and sufficient condition, in terms of the p_n, for $X_n \to 0$ in probability.

b) Find and prove a necessary and sufficient condition, in terms of the p_n, for $X_n \to 0$ almost surely.

HINT: consider conditions such as $p_n \to 0, \limsup p_n < 1, \sum_n p_n^2 < \infty, \sum_n p_n < \infty$.

2. Suppose that $f(x)$ is a continuous function on $[0,1]$, $0 \le f(x) \le 1$, and let $J = \int_0^1 f(x)dx$. Let (X_i, Y_i), $i = 1, 2, \ldots$ be a sequence of independent uniformly distributed over $[0,1]$ random variables. Let $I_i = I_{\{f(X_i) \ge Y_i\}}$ be the indicator of the event $\{\omega : f(X_i) \ge Y_i\}$, and let $J_n = n^{-1} \sum_{i=1}^n I_i$ and $J_n^* = n^{-1} \sum_{i=1}^n f(X_i)$, $n = 1, 2, \ldots$.

a) Why $\lim_{n\to\infty} J_n = \lim_{n\to\infty} J_n^* = J$ with probability 1?

b) Show that the mean square error of J_n^* does not exceed the mean square error of J_n : $E[(J_n^* - J)^2] \le E[(J_n - J)^2]$. For what continuous functions $f(x)$ both errors coincide?

c) Use the CLT to find n such that $P(|J_n - J| \le 0.01) = 0.9$, independently of f.

3. a) Give the definitions of the convergence in probability and convergence in distribution.

b) Let X be a Bernoulli random variable taking values 0 and 1 with equal probability $\frac{1}{2}$. Let $X_1, X_2, \ldots$ be identical random variables given by $X_n = X$ for all n and let $Y = 1 - X$.

Does X_n converges to Y in probability? Does X_n converges to Y in distribution?

c) Prove that if a sequence of random variables Y_n converges to Y in probability, then it converges to X in distribution.