Real analysis, Graduate Exam Fall 2006

Answer all four questions. Partial credit will be given to partial solutions.

1. Find necessary and sufficient conditions for a subset $X \subset \mathbb{R}$ to belong to the σ-algebra generated by all one-point subsets of $\mathbb{R}$.

2. Let (X, μ) be a measure space. Which of the following implications are true?

a. $\mu(X) < \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

b. $\mu(X) = \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

c. $\mu(X) < \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

d. $\mu(X) = \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

Give proof or counter-example in each case.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} e^{-(x-y)} & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -e^{-(y-x)} & \text{if } x < y. \end{cases}$$

a. Is f Lebesgue integrable?

b. Is it true that

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\,dx \right) dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\,dy \right) dx \ ?$$

4. Let $f \in L^1(\mathbb{R})$. Show that for each $n = 1, 2, 3, \ldots$, the function

$$f_n(x) = f(x)(\sin x)^n$$

also belongs to $L^1(\mathbb{R})$ and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0.$$