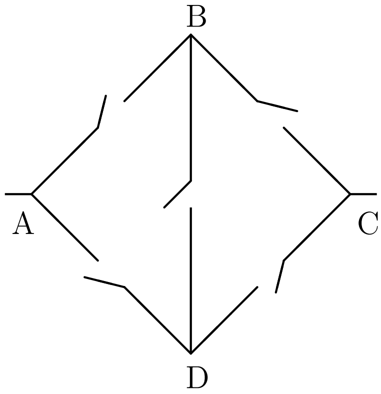

FIGURE 1. A random connection

Fall 2006 Qualifying exam, Math 505a

Do all three problems, attempt all parts

Problem 1. Let X, Y, and Z be independent standard normal random variables.

(a) Show that $X^2 + Y^2$ and $\frac{X}{\sqrt{X^2+Y^2}}$ are independent.

(b) Show that

$$\frac{X + YZ}{\sqrt{1 + Z^2}}$$

is standard normal (Hint: condition on Z).

Problem 2. On Figure 1, each of the five connections can be open or closed independently of other connections. The probability to have a specific connection closed is p.

(a) Find the probability that there is a path of closed connections from A to C.

(b) Find the conditional probability that the connection along the diagonal BD is closed given that there is a path of closed connections from A to C.

Problem 3. Let S_n a random walk on $\mathbb{Z}$, with $S_0 = 0$. Let $\tau_0 = \inf\{n > 0 \colon S_n = 0\}$, the hitting time of 0.

(a) Show that

$$1 = \sum_{m=0}^{n} P_0(S_{n-m} = 0)\, P_0(\tau_0 > m).$$

(Hint: Condition according to the last time, that the chain will visit 0, before time n.)

(b) Assume further that S_n is *simple* random walk, that is, steps are plus one or minus one with probability one-half each. Assume also that n is even. The first term in the sum, indexed by $m = 0$, is simply $P(S_n = 0)$. Give a simple expression a_n which is asymptotic to this, that is, such that the ratio $a_n/P(S_n = 0)$ is close to 1 for large even n.

c) Continuing (b), the last term in the sum, indexed by $m = n$, is simply $P(\tau_0 > n)$. Give a simple expression b_n which is asymptotic to this.