PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Fall 2022

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider a solution u to the nonlinear equation

$$\begin{cases} -\Delta u = \lambda u^2(1-u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ and Ω is a bounded domain with smooth boundary. Prove that $0 \le u \le 1$.

2. Let $u(x,t)$ be a solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = g(x), \ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^3, \end{cases}$$

where $g, h \in C^\infty(\mathbb{R}^3)$ and are supported in the ball $B(0,R)$.

(a) Use the Kirchoff formula to show that

$$|t \cdot u(x,t)| \le C\left(2 + \frac{1}{t}\right) |\partial B(x,t) \cap B(0,R)|$$

for all $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, where $|\cdot|$ denotes two-dimensional Lebesgue measure.

(b) Show that for some $t_0 > 0$ the measure $|\partial B(x,t) \cap B(0,R)|$ can be bounded by a constant independent of $t \ge t_0$, and conclude that

$$|u(x,t)| \le \frac{C}{t}$$

for all $(x,t) \in \mathbb{R}^3 \times (0,+\infty)$, where $C > 0$ is a constant.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume $u(x,t) \ge 0$ is a function with $u \in C^2(\overline{\Omega} \times [0,+\infty))$, which solves the heat conduction equation with heat loss due to radiation

$$\begin{cases} (\partial_t - \Delta)u = -u^4 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

prove that we can find a constant C *independent of* the initial data $u(0)$, such that

$$E(1) := \int_\Omega u(x,1)^2 \, \mathrm{d}x \le C.$$