

Numerical Analysis Qualifying Exam Problems - Spring 1996

There are seven problems in this exam. You should solve any five (and no more than five) problems.

Problem 1. Linear systems.

Let $A = \begin{pmatrix} 1 & -1 \\ 1 & -1.00001 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & -1 \\ -1 & 1.00001 \end{pmatrix}$.

(a.) Compute the ratio of maximum to minimum eigenvalue for both matrices. Is matrix A well- or ill-conditioned? Make extensive comments.

(b.) Give a posteriori estimate of the relative error in solving $Ax = b$ and $Bz = b$, respectively, where $\|b\|_2 = 1$.

Problem 2. Least squares.

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and x a fixed vector with $\|x\|_2 = 1$. Show that $\lambda_R = x^T Ax$ minimizes the function $f(\lambda) = \|Ax - \lambda x\|_2$.

Problem 3. Interpolation.

(a.) Find the quadratic interpolant to $f(x) = e^x$ on $\{x_i\} = \{0, \frac{1}{4}, 1\}$.

(b.) Using Hermite interpolation determine the fourth degree polynomial $p(x)$ for which:

$$p(0) = p'(0) = 0, \quad p(1) = p'(1) = 1, \quad p(2) = 1.$$

Problem 4. Ordinary differential equations.

Consider the initial value problem

$$y' = f(y)$$

$$y(0) = y_0,$$

where $y_0 \in \mathbb{R}$ is given, and the numerical integration scheme given by

$$\frac{y_{n+1} - y_{n-1}}{2h} = f(y_n), \quad n = 1, 2, \dots, \quad (1)$$

$$y_1 = y_0 + hf(y_0), \quad (2)$$

where $h > 0$.

(a). Let $f(y) = ay$, where $a \in \mathbf{R}$ and use the definition of y_1 given in (2) to find the solution to the resulting second order linear difference equation (1) in terms of y_0 .

(b). Use the fact that $h \ll 1$ implies $1 + h^2 a^2 \approx 1$ and the solution found in part (a) to show that for n large and h small with $nh = t$ we have

$$y_n \approx c_1 e^{at} + c_2 (-1)^n e^{-at},$$

for some constants c_1 and c_2 .

(c). If $a < 0$ would this be a good choice of scheme? Why or why not?

Problem 5. Nonlinear equations.

Consider the iteration

$$x_{n+1} = \frac{x_n^2 + c}{2x_n}, \quad n = 0, 1, 2, \dots,$$

with $x_0 \in \mathbf{R}$ given. Find an interval $I \subset \mathbf{R}$ for which $x_0 \in I$ guarantees that $\lim_{n \rightarrow \infty} x_n$ exists. What is the value of $\lim_{n \rightarrow \infty} x_n$?

Problem 6. Numerical Integration.

(a). Derive the trapezoidal rule for integrating a function $f(x)$ over the interval $a \leq x \leq a + h$ and express the error in terms of h and a derivative of f .

(b). Derive the composite trapezoidal rule to integrate f over an interval (a, b) with step-size h , and express the error in terms of h . As $h \rightarrow 0$ do the approximations converge to the integral? How do you use the error estimate to obtain an approximation to four decimal digits?

(c). State Simpson's Rule for integrating $f(x)$ over $(a, a + h)$ and express the error in terms of h . State the composite Simpson's Rule for the interval (a, b) with step-size h and express the error in terms of h .

Problem 7. Spline interpolation.

(a). Define the cubic spline, $S(x)$, interpolating the set of points (x_i, y_i) , $0 \leq i \leq N$.

(b). Derive a method for computing $S(x)$.

(c). Let $f(x_i) = y_i$, $0 \leq i \leq N$. Let $S(x)$ be the cubic spline in part (a). Suppose f is Lipschitz continuous on an interval (a, b) containing the nodes x_i . With appropriate end-conditions for $S(x)$, express the error $\|f - S\|_\infty$ in terms of the maximum of the mesh sizes $h_i = x_i - x_{i-1}$. As $\max h_i \rightarrow 0$ do the splines converge to f ? Compare this result with the sequence of Lagrange interpolating polynomials over equally-spaced grids.

Numerical Analysis Qualifying Exam Problems - Spring 1997

There are eight problems in this exam. You should solve any five (and no more than five) problems.

Problem 1. The trapezoidal rule for integration is

$$(Tr) \quad \int_a^{a+h} f(x) dx = \frac{h}{2}(f(a) + f(a+h)).$$

- Derive (Tr) using the interpolating polynomial.
- Prove that the error in (Tr) is $-\frac{h^3}{12}f^{(2)}(\alpha)$, $a < \alpha < a+h$.
- Extend (Tr) to obtain a composite numerical quadrature formula for $\int_a^b f(x) dx$ with uniform spacing $h = (b-a)/n$ and grid points $x_i = a+ih$, $i = 0, 1, \dots, n$.
- Show that the error in the composite trapezoidal rule in part (b) is $(-1/12)(b-a)h^2 f''(\xi)$, $a < \xi < b$.
- Use (Tr) to derive a numerical finite-difference formula to solve an ordinary differential equation $dy/dt = f(t, y)$ with given initial condition $y(0) = y_0$. Discuss the order of accuracy, both local and cumulative.

Problem 2. a. Use Taylor's formula to derive the second-order difference formula

$$(D) \quad f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2).$$

- Consider the one-dimensional heat equation $u_t = u_{xx}$, (H) with boundary conditions, $u(0, t) = U_0$, $u(1, t) = U_1$, (B) and initial conditions, $u(x, 0) = f(x)$. (I)

Use formula (D) to replace u_{xx} and the trapezoidal rule (Tr) in problem 1 to replace u_t so as to obtain a finite-difference formula to solve (H), (B), (I) on the rectangle $0 \leq x \leq 1$, $0 \leq t \leq T$. Discuss the resulting algorithm.

- Show that the accuracy of the method in part b. (so-called Crank-Nicholson method) is $O(\Delta x^2) + O(\Delta t^2)$.

Problem 3. The following two schemes are used for solving $y'(t) = f(t, y)$:

a. midpoint rule $\frac{1}{2}y_{n+1} - \frac{1}{2}y_{n-1} = hf(t_n, y_n)$,

b. backward differentiation formula $\frac{3}{2}y_{n+1} - 2y_n + \frac{1}{2}y_{n-1} = hf(t_{n+1}, y_{n+1})$.

Discuss their stability, zero-stability, and practical aspects of usage.

Problem 4.

a. Prove that a matrix $A \in C^{n \times n}$ is normal, $A^H A = A A^H$, if and only if A is unitary diagonalizable, that is, there exists a unitary matrix $U \in C^{n \times n}$ such that $U^H A U = D = \text{diag}(\lambda_1, \dots, \lambda_n)$.

b. Let $A \in C^{n \times n}$ be defined by: $A_{k,k} = 2k$, $A_{k,l} = i(k+l)$, $k \neq l$, $k, l = 0, 1, \dots, n-1$, $i = \sqrt{-1}$. Show that A is normal.

Problem 5.

a. Derive a system of nonlinear equations to find the abscissas x_0, x_1 and weights ω_0, ω_1 for a two-point Gauss quadrature rule for $\int_{-1}^1 f(x) dx$.

b. Set up Newton iterations for this system, and compute the first iterate starting with the initial guess $(x_0, x_1, \omega_0, \omega_1)^T = (-1, 1, 1, 1)^T$.

Problem 6.

a. Give a necessary and sufficient condition for x to be a solution to $\min_x \|Ax - b\|_2$, and interpret this geometrically. When is the least squares solution x unique?

b. Prove the above statements.

Problem 7.

a. Define Chebyshev polynomials $T_n(x)$ for $x \in [-1, 1]$ and show that $x_k = \cos \frac{(2k+1)\pi}{2n}$, for $k = 0, 1, \dots, n-1$ are the zeros, and $\bar{x}_k = \cos \frac{k\pi}{n}$, for $k = 0, 1, \dots, n$ are the extrema of $T_n(x)$.

b. Show that of all n^{th} degree polynomials with leading coefficient 1, the normalized Chebyshev polynomials $2^{1-n} T_n(x)$ have the smallest norm in the interval $[-1, 1]$.

Problem 8. Consider the following 3 by 3 matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 4 \\ -3 & 9 & 6 \\ 1 & -1 & -2 \end{bmatrix}.$$

Estimate the eigenvalues of the matrix $C = A + 0.01B$. How many distinct eigenvalues of C are there?

Numerical Analysis Qualifying Exam Problems - November 1997

There are six problems in this exam.

Problem 1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

- a. Apply one iteration of the power method with any nonzero starting vector.
- b. Apply one iteration of the QR algorithm to A .
- c. Determine the exact eigenvalues of A and the eigenspace corresponding to the largest eigenvalue.
- d. Compare your answers in a, b and c.

Problem 2. Let $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

where

ϵ is a small positive number.

- a. Determine the singular values of A and thus the least squares solution x exactly.
- b. Suppose that ϵ is sufficiently small that $1 + \epsilon^2$ gets rounded off to 1 on your computer. Determine the eigenvalues of the calculated $A^T A$ and the ensuing solution x .
- c. Use the Gram-Schmidt orthogonalization to compute $A = QR$ and the resulting solution x even in the presence of the above round off.

Numerical Analysis Qualifying Exam Problems - Fall 1997

Problem 3. Consider the Adams Moulton corrector formula

$$y_{n+1} = \tag{1} y_n + \frac{h}{24} (9f(x_{n+1}, y_{n+1}) + 19f(x_n, y_n) - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2}))$$

for the initial value problem

$$\begin{aligned} y' &= f(x, y) \\ y(0) &= y_0. \end{aligned}$$

(a). If the Adams Moulton corrector given in (1) is applied to the ODE

$$y' = ay, \tag{2}$$

find the resulting difference equation.

(b). Let $\lambda_i(h)$, $i = 1, 2, 3$ be the characteristic values for the difference equation found in part (a), and show that

$$\lim_{h \rightarrow 0} \lambda_1(h) = 1, \quad \lim_{h \rightarrow 0} \lambda_2(h) = 0, \quad \lim_{h \rightarrow 0} \lambda_3(h) = 0.$$

(c). Show that if $\lambda_1(h) = 1 + \sum_{j=1}^{\infty} a_j h^j$, then $a_1 = a$.

(d). Show that $\lim_{h \rightarrow 0, n \rightarrow \infty, nh=t} \lambda_1(h)^n = e^{at}$.

(e). What can you say about convergence of the Adams Moulton scheme for the equation given in (2).

Problem 4. Let α , β , and γ be three points in the complex plane, and let T_0 be the triangle formed by joining them with three line segments. Consider the following iteration. Let T_n be the triangle formed by joining the mid-points of the three sides of triangle T_{n-1} .

(a). Determine a representation $\tau_n \in \mathbb{C}^3$ for triangle T_n , and determine a real matrix M such that

$$\tau_n = M\tau_{n-1}. \tag{3}$$

(Hint: $\tau_0 = [\alpha, \beta, \gamma]^T$.)

(b). Using the formulation given in (3), show that the iteration converges to the centroid of T_0 . (Hint: Diagonalize the matrix M .)

(c). Generalize the results of (a) and (b) to the case where $\alpha_1, \alpha_2, \dots, \alpha_m$ are m points in the complex plane, and T_0 is the figure obtained by connecting them sequentially in some order by line segments with the last point being joined to the first. Then let T_n be the figure obtained by connecting the midpoints of the line segments in figure T_{n-1} .

Problem 5. Polynomial Approximation

Let $f(x)$ be a real-valued function defined for x in a bounded set S on the real line.

- (a) Define the best approximating polynomial, π_n , of degree $\leq n$ for f on S . Also, define the least squares polynomial p_n .
- (b) State the necessary and sufficient condition for π_n on a subset of $n + 2$ points.
- (c) Determine π_n and p_n for the following S, f, n .
 - (i) $S = \{-1, 1\}, f(-1) = 0, f(1) = 1, n = 0$;
 - (ii) $S = \{-1, 0, 1\}, f(-1) = f(1) = 1, f(0) = 0, n = 1$;
 - (iii) $S = \{-1, 0, 1\}, f(-1) = -2, f(0) = 1, f(1) = 0, n = 1$;
 - (iv) $S = [0, 1], f(x) = 1 - x^2, n = 1$.

(Hint: Draw a graph to see that 0 and 1 are extremum points of the residual function $r(x) = f(x) - \pi_1(x)$. Also there is one interior extremum a where $r'(a) = 0$. Use part (b).)

- (d) Specify the Remes single-exchange algorithm for π_n .

Problem 6. Numerical Integration

Let f be Riemann integrable on $[a, b]$.

- (a) Define the Lagrange interpolating polynomial $L_n(x)$ of degree $\leq n$ on $n + 1$ points $(x_i, f(x_i)), i = 0, \dots, n$. State the error formula for $f(x) - L_n(x)$.
- (b) Use $L_n(x)$ to obtain a general quadrature formula to approximate $\int_a^b f(x)dx$. Specify its order of accuracy.
- (c) Derive the composite trapezoidal rule and its order of accuracy over $[a, b]$.
- (d) State Simpson's 1/3 Rule and specify its accuracy.

Problem 3. (Non-linear equations)

(a) Let A be an $n \times n$ matrix and let $b = (b_1, \dots, b_n)$ be the n -dimensional vector with all components $b_i = 1$. Consider the equations

$$(1) \quad (A + \alpha I)x = b,$$

$$(2) \quad x^T x = 1.$$

Take $A = \text{diag}(1, 2, 3)$, a diagonal matrix of order $n = 3$. Determine α to one significant digit so that (1) is satisfied exactly and (2) approximately.

(b) Set up the Newton iteration formula for solving $z^2 + 1 = 0$ using real arithmetic.

(c) State a convergence theorem for Newton's method.

(d) Discuss possible modifications to Newton's method (or the secant method) so that the resulting algorithm has global convergence provided that upper and lower bounds of the root are known.

Problem 4. (Numerical Stability, ill-conditioning)

(a) Give a definition of numerical stability of a numerical algorithm.

(b) Discuss the numerical stability of algorithms for solving a system of linear algebraic equations. Include Gauss elimination with partial and complete pivoting in your discussion. Discuss the condition number of the system and its effect on numerical stability.

(c) Sketch a proof that the QR algorithm for determining eigenvalues is stable.

Problem 5. (ODE) Consider the initial-value problem

$$(IVP) \begin{cases} \frac{dx}{dt} = f(t, x), \\ x(t_0) = x_0, \end{cases}$$

where x is an n -dimensional vector.

(a) Use numerical quadrature to derive the Euler method for solving (IVP) and its local truncation error. Do the same for the trapezoidal rule. Explain how these two methods can be organized into a predictor-corrector method. How would you determine the step-size, h ?

(b) Consider the general explicit quadrature method of order $n + 1$,

$$x_j = x_{j-1} + h \sum_{i=0}^n a_i f(t_{j-i-1}, x_{j-i-1}), \quad j > n,$$

with local truncation error $R_n(j) = O(h^{n+2})$. Let $x(t)$ be the exact solution of (IVP) and $e_j = x(t_j) - x_j$. Prove that $\|e_j\| \rightarrow 0$ as $h \rightarrow 0$ under suitable conditions on f and the starting values x_0, x_1, \dots, x_n .

Problem 6. (Stiff ODE)

Consider the linear system of differential equations

$$(L) \quad \frac{dx}{dt} = Ax,$$

where A is a constant $n \times n$ matrix.

- (a) When would system (L) be considered stiff?
 (b) Consider the case $n = 1$,

$$(L1) \quad \frac{dx}{dt} = \lambda x.$$

Apply the Euler method to (L1) and determine its region of absolute stability. How should the step-size h be chosen for the case $\lambda < 0$?

(c) Apply the backward Euler method to (L1) and determine its region of absolute stability. How should the step-size h be chosen for the case $\lambda < 0$ and $|\lambda|$ large? How should h be chosen for the case $\lambda > 0$ and $|\lambda|$ large?

Problem 7. (BDF methods)

Consider the general linear multi-step (LMS) method for solving $dx/dt = f(t, x)$, as given by

$$(LMS) \quad \sum_{j=0}^K a_j x_{n+1-j} = h \sum_{j=0}^K b_j f(t_{n+1-j}, x_{n+1-j}),$$

for $n \geq K - 1$, with x_0, x_1, \dots, x_{K-1} as starting values.

- (a) Define consistency for LMS.
 (b) Write the characteristic polynomial $p(\xi)$ for the left side. State a necessary and sufficient condition for a consistent LMS method to be stable and convergent.
 (c) Which choices of the b_j and a_j define LMS to be a BDF method? For which values of K are the BDF methods stable and convergent?
 (d) Consider the test case $dx/dt = \lambda x$. For the general LMS, write the polynomial $q(\xi)$ for the right side and obtain the characteristic polynomial equation $p(\xi) - h\lambda q(\xi) = 0$ for this case. Define the region of absolute stability of the LMS. Write $q(\xi)$ for the BDF methods.
 (e) Sketch the regions of absolute stability of the BDF methods. What feature of these regions makes the BDF methods good stiff solvers?

Problem 8. (Least Squares)

- a) Give a necessary and sufficient condition for x to be a solution to $\min x : \|Ax - b\|^2$, and interpret this geometrically. When is the least squares solution x unique?
- b) Consider fitting $f(x) = \sqrt{x}$ with $L(x) = ax + b$ on the interval $[\frac{1}{4}, 1]$ in the least squares sense. Set up the normal equations (compute the entries for the matrix and the right hand side) without actually solving them.

Numerical Analysis Qualifying Exam Problems - Fall 1998

There are seven problems in this exam. You should solve any five (and no more than five) problems.

Problem 1. Differential equations

Consider the *midpoint method*: $y_{n+1} = y_{n-1} + 2hf_n$.

- Find its region of (absolute) stability for the test problem $y' = qy$, $y(0) = 1$, where $q \in C$.
- Let $y' = Ay$, where A is a skew-symmetric matrix: $A^T = -A$. Show that the use of the midpoint method for such a problem makes sense.
- How the result from (b.) can guide you in solving a related nonlinear problem (a simplification of the predator-prey):

$$x' = ax(1 - y), \quad y' = -ay(1 - x)$$

(Hint: investigate the system at the critical points u^* : $f(u^*) = 0$, where $u = (x, y)^T$ and $u' = f(u)$ given above).

Additionally, what step size, h would you use?

Problem 2. Interpolation

- Define a cubic spline. Compare the number of conditions and the number of unknown parameters.
- Derive the *Not-a-knot* spline that enforces continuity of the 3rd derivative at the second knot (at both ends of the interval).

Problem 3. Least squares

- Find the least squares fit $y(x) = a + \frac{b}{x}$ to the data given by $f(x) = e^{-x}$ on $x = \{.1, .5, 1\}$,
- Formulate in detail (without actually solving) how to solve a nonlinear least squares fit $y(x) = ae^{bx}$ to a given data $f(x)$ (with some initial guess a_0, b_0).

Problem 4. Polynomial Approximation

- State the Weierstrass theorem on polynomial approximation of a continuous function f on a finite interval $[a, b]$.
- Define the best approximating polynomial (BAP) $\pi_n(a, b)$ degree $\leq n$ for f on $[a, b]$. Prove that $\pi_n(a, b)$ exists. Is it unique?
- Describe the steps of the Remes single-exchange algorithm for computing $\pi_n(a, b)$.
- Define the least squares polynomial p_n of degree $\leq n$ for f on a set $E \subset [a, b]$. Prove that p_n exists and is unique. What numerical difficulty arises in computing p_n as $n \rightarrow \infty$?

(e) Consider the set $E = \{-1, 0, 1\}$. Let $f(-1) = f(1) = 1$ and $f(0) = 0$. Find π_0 , the BAP of degree 0 for f on E . Prove your answer. Find p_0 , the least squares polynomial of degree 0 for f on E . Find π_1 for f on E and prove your answer. Find π_n for $n \geq 2$. Find p_1 , the least squares polynomial of degree ≤ 1 .

Problem 5. Numerical Integration

Let $f \in C^{(n+1)}[0, 1]$. Let $E\{x_i : i = 0, \dots, n\}$ be $n + 1$ points in $[0, 1]$.

(a) Give a formula for $L_n(x)$, the Lagrange interpolating polynomial of f on E .

(b) Give a formula for $r(x) = f(x) - L_n(x)$ in terms of $f^{(n+1)}$ and $\phi_{n+1}(x) = \prod_{i=0}^n (x - x_i)$.

(c) Consider the quadrature functional $F_n f = \int_0^1 L_n(x) dx$.

Express it in the form $F_n f = \sum_{i=0}^n a_i f(x_i)$ and give the formulas for the a_i .

(d) Let $R_n f = \int_0^1 f(x) dx - F_n f$. Express $R_n f$ in terms of $f^{(n+1)}$ and ϕ_{n+1} .

(e) Consider the points $x_i = a + ih$, $i = 0, \dots, n$. Obtain a bound for $R_n f$ in part (d) in terms of h .

(f) Derive the trapezoidal rule and its error formula.

Problem 6. Eigenvalue problems

(a) Consider the polynomial $p(x) = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$. Show that if $\zeta \in \mathbf{C}$ is a root of p (i.e. $p(\zeta) = 0$), then $\zeta \in \cup_{k=1}^n C_k$, where the circles are $C_1 = \{z \in \mathbf{C} : |z + a_1| \leq 1\}$, $C_j = \{z \in \mathbf{C} : |z| \leq 1 + |a_j|\}$, $j = 2, 3, \dots, n-1$, and $C_n = \{z \in \mathbf{C} : |z| \leq |a_n|\}$.

(b) Show that if $\zeta \in \mathbf{C}$ is a root of p , then $\zeta \in C_p = \{z \in \mathbf{C} : |z| \leq 1 + \max_{1 \leq j \leq n} |a_j|\}$.

(c) Let $p(x) = x^4 + 8x^3 - 8x^2 - 200x - 425$. Show that if ζ is a root of p , then $\zeta \in \{z \in \mathbf{C} : |z + 4| \leq 9\}$.

Problem 7. Iterative methods: Let A be an $n \times n$ matrix.

(a) Define what it means for A to be *diagonally dominant*.

(b) Describe the Jacobi Iteration Method for solving the linear system $Ax = b$, and provide an explicit formula for the Jacobi iterates, x_k , $k = 0, 1, 2, \dots$

(c) Show that if A is diagonally dominant, then A is non-singular.

(d) Show that if A is diagonally dominant, then the Jacobi iterates, x_k , $k = 0, 1, 2, \dots$ are well defined, and that $\lim_{k \rightarrow \infty} x_k = x$, for any initial guess $x_0 \in \mathbf{R}^n$, where x is the unique solution to $Ax = b$.

Numerical Analysis Qualifying Exam Problems - Spring 1999

There are seven problems in this exam. You should solve any five (and no more than five) problems.

Problem 1. Iterative methods

a. Define the SOR method for solving $Ax = b$.

b. Formulate the convergence theorem for optimal SOR iterations.

c. Let $A : A(i, j) = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{for } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$ $i, j = 1, \dots, n$. Eigenvalues

of A are $\lambda_i = 2 - 2\cos(\pi ih) = 4\sin^2(\frac{\pi ih}{2})$, where $h = \frac{1}{n+1}$. Estimate the number of iterations required to reduce the initial error by a given factor τ as a function of n for a. Jacobi, b. Gauss-Seidel, c. optimal SOR. (use low order polynomial approximation of $\sin(x)$, $\ln(1+x)$, etc., when applicable).

Problem 2. ODEs

Find the most accurate one-step LMM $y_{n+1} + a_0 y_n = h(b_1 f_{n+1} + b_0 f_n)$.

a. What is its order? Is it zero stable? b. What is its error constant?

c. What is its region of absolute stability?

Problem 3. Eigenproblems

a. Let all of the row sums of an n by n matrix A have the same value α . Show that α is an eigenvalue of A ; what is the corresponding eigenvector?

b. What are the eigenvalues of the Householder transformation $H = I - 2\frac{vv^T}{v^T v}$, where v is a nonzero vector?

c. Give an example of a 2×2 matrix A and a nonzero starting vector x_0 such that the power method fails to converge to the eigenvector corresponding to the dominant eigenvalue of A .

Problem 4. Quadrature

a. Define Newton-Cotes quadrature rules for definite integrals.

b. Derive a formula

$$\frac{1}{\sqrt{2h}} \int_0^{2h} \frac{f(x)}{\sqrt{x}} dx \approx c_0 f(0) + c_1 f(h) + c_2 f(2h)$$

which is exact when $f(x)$ is a second degree polynomial.

c. The following values of an integral were computed using the Trapezoidal rule with step sizes h , $h/2$, $h/4$ and $h/8$:

$$T_0 = .7586780454, T_1 = .7687573650, T_2 = .7712621711, T_3 = .7718874437.$$

Use the repeated Richardson extrapolation to compute an improved value of the integral to 8 decimal places.

Numerical Analysis PhD Exam May 1999

Linear Least Squares

51. Let A be an $m \times n$ real matrix and let b be an m -dimensional real vector. Consider the equation

$$(*) \quad Ax = b.$$

- Define $\text{rank}(A)$, $\text{range}(A)$ and $\text{nullity}(A)$. State a necessary and sufficient condition that $(*)$ have a solution.
- Define the least squares solution, x^* , of $(*)$. Does x^* always exist? Prove your answer.
- Discuss the uniqueness of x^* . Define the pseudo-inverse, A^\dagger , of A . Does it always exist? What are its dimensions? What are the properties of $A^\dagger b$?
- Describe a numerical algorithm for computing x^* .

Polynomial Approximation

64. Let f be a real function defined on the interval $[0, 1]$.

- State the Weierstrass theorem on polynomial approximation of f .
- Define the best approximating polynomial (BAP) π_n of degree $\leq n$ for f on $[0, 1]$. Prove that π_n exists for any bounded f .
- State a necessary and sufficient condition for a polynomial to be the BAP of degree $\leq n$ for f on a set of $n+2$ points in $[0, 1]$. Use it to obtain π_0 for $f(0) = 0$, $f(1) = 1$ on the set $\{0, 1\}$ and π_1 for $f(0) = 0$, $f(1/2) = -1/2$, $f(1) = 1$ on the set $\{0, 1/2, 1\}$. Describe the steps of the single-exchange algorithm for π_n on $[0, 1]$.
- Define the least squares polynomial p_n of degree $\leq n$ for f . Suppose q_0, \dots, q_n are given orthogonal polynomials of degrees $0, 1, \dots, n$ on $[0, 1]$. Describe how to compute p_n using the orthogonal polynomials.
- Let $f(x_i)$ be given on a set E of N equally-spaced points x_i in $[0, 1]$. Describe how to compute the least-squares polynomial $p(x) = a_0 + a_1x + \dots + a_nx^n$ of degree $\leq n$ on the set E by solving a linear least-squares problem $Aa = b$, where $a = (a_0, \dots, a_n)^T$. Write the matrix A and discuss its condition number for $n > 4$. Write the normal equations and discuss the condition number of $A^T A$. Would you use this method to compute p accurately? Explain your answer. If the data are noisy, how else can you smooth them using polynomials?

Problem 7. Nonlinear equations.

- Let $g: V \rightarrow V$ be a real mapping of an n -dimensional space V onto itself. State a contractive mapping theorem for g .
- How is the theorem applied to finding a fixed-point of g ? What is the rate of convergence of the iterates?
- Let $f: V \rightarrow V$, where V is n -dimensional as above. Define an iteration to solve $f(x) = 0$ using the derivative $f'(x)$ to construct a function g which is locally contractive. Sketch a proof of local contraction. What is the rate of convergence?
- Describe a computer algorithm to implement the iteration in (c).
(The iteration formula is not sufficient detail.)
- Illustrate the use of the algorithm to compute the square root of a positive number. What "tricks" would you use to make this a practical subroutine for the square root function?

Numerical Analysis Qualifying Exam Fall 1999

Please select 5 out of total 6 problems to solve and mark clearly in the table below which of the 5 problems that you have selected. Only the marked problems will be graded.

Last Name:

First Name:

Social Security Number:

Problem	Mark Selection	Grade
1		
2		
3		
4		
5		
6		
Total		

Problem 1.

- (a) In your "Star War" simulation project run with a machine precision of about $1e-16$ you require about 10 decimals of accuracy in the results. For what range of values of the condition number $\kappa(A)$ you would call a matrix ill-conditioned?
- (b) Consider a family of matrices $A \in R^{n \times n}$ defined by $A = I + vv^T$, $v \in R^n$. Find the value of the condition number $\kappa_2(A)$ and then choose appropriate v 's such that
- (b.1) $\kappa_2(A)$ does not grow as n increases, i.e, A remains well-conditioned,
- (b.2) $\kappa_2(A)$ grows fast as n increases, i.e, A rapidly becomes ill-conditioned.

Problem 2.

- (a) Consider the problem: find $\min_{x \in S} \|x\|_2$, $S = \{x \in R^n, \|b - Ax\|_2 = \min\}$, where $A \in R^{m \times n}$ and $\text{rank}(A) = r \leq \min(m, n)$. Show that using SVD this problem always has a unique solution.
- (b) Compute such a minimum (by any method) for $A \in R^{3 \times 2} : A_{i,j} = 1$, for all i, j and $b = (1, 2, 3)^T$.

Problem 3.

Consider the following polynomial $p(\lambda) = (\lambda - 1)^3 \lambda^3$.

- (a) How many non-similar matrices can you find whose characteristic polynomial is p ? Please give an example of such a collection of matrices.
- (b) Suppose A is a full matrix with $\det(A - \lambda I) = p(\lambda)$ and A has exactly one unit eigenvector for each of its eigenvalues. When the power method is used to compute the largest eigenvalues of A , will the result be the correct value?

Problem 4.

The divided difference of a function is defined recursively as:

$$f[x_0] = f(x_0), \quad f[x_n, \dots, x_0] = \frac{f[x_n, \dots, x_1] - f[x_{n-1}, \dots, x_0]}{x_n - x_0},$$

where $\{x_k\}_{k=0}^n$ are distinct points. Consider the following function:

$$f_t(x) = \begin{cases} (x-t)^m, & x \geq t, \\ 0, & x < t. \end{cases}$$

Suppose that the sequence of grid points $x_0 < x_1 < x_2 < \dots < x_{m-1} < x_m$. Show that the function $s(t) = f_t[x_m, \dots, x_0]$ has the following properties:

- $s(\cdot)$ is in C^{m-1} .
- $s(t)$ equals to zero for t outside of the interval $[x_0, x_m]$.
- Over each of the intervals $[x_k, x_{k+1}]$ for $k = 0, \dots, m-1$, s is a polynomial of degree m .

Problem 5.

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be twice continuously differentiable with $f(\alpha) = 0$ and $f'(x) \neq 0$, $x \in (\alpha - \epsilon, \alpha + \epsilon)$, some $\epsilon > 0$.

- Write down the iteration formula for Newton's method for determining a solution to the equation $f(x) = 0$.
- Let x_n denote the n -th Newton iterate and set $e_n = x_n - \alpha$. Use Taylor's formula with remainder to show that

$$\lim_{n \rightarrow \infty} \frac{e_{n+1}}{e_n^2} = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)},$$

and hence that Newton's method yields quadratic convergence to simple roots.

- Describe how Newton's method could be used to locate local extrema of a function g .

(Problem 5 continues on next page)

Problem 5. (Continued)

(d) Let $g(x) = \sqrt{1+x^2}$, and show that the Newton iterates $\{x_n\}_{n=0}^{\infty}$ for finding the minimum of g have the property that

(i) If $|x_0| < 1$, then $g(x_{n+1}) < g(x_n)$, $n = 0, 1, 2, \dots$, and $\lim_{n \rightarrow \infty} x_n = 0$.

(ii) If $|x_0| > 1$, then $g(x_{n+1}) > g(x_n)$, $n = 0, 1, 2, \dots$, and $\lim_{n \rightarrow \infty} |x_n| = +\infty$.

Problem 6.

Consider the problem of plotting a circle of radius $r > 0$ in the plane centered at the origin, $\{x = r \cos \theta, y = r \sin \theta\}$, $\theta \in [0, 2\pi]$, by numerically solving the initial value problem (IVP) given by

$$\frac{dx}{d\theta} = -y, \quad 0 < \theta < 2\pi,$$

$$\frac{dy}{d\theta} = x, \quad 0 < \theta < 2\pi,$$

$$x(0) = r, \quad y(0) = 0.$$

- (a) Show that the solution $(x(\theta), y(\theta))$, $\theta \in [0, 2\pi]$ to (IVP) is in fact the desired circle.
- (b) Write down the iteration formulas which result from using Euler's method with stepsize $h = \frac{2\pi}{n}$ to discretize (IVP).
- (c) Show that in fact Euler's method is a poor choice for a numerical integration scheme since the resulting iterates lie on a spiral.
- (d) Show that the implicit iteration formulas given by

$$x_{n+1} = x_n - hy_n, \quad n = 0, 1, 2, \dots, n,$$

$$y_{n+1} = y_n + hx_{n+1}, \quad n = 0, 1, 2, \dots, n,$$

are a better choice since in this case, if the effects of round off error are ignored, the resulting iterates will lie on a closed curve in the plane.

Numerical Analysis Qualifying Exam Spring 2000

Please select 5 out of total 7 problems to solve and mark clearly in the table below which of the 5 problems that you have selected. Only the marked problems will be graded.

Last Name:

First Name:

Social Security Number:

Problem	Mark Selection	Grade
1		
2		
3		
4		
5		
6		
7		
Total		

Problem 1. (Iterative methods)

- (a) Formulate the main theorem for SOR iterations that allows one to find the optimal value of the parameter, ω and the corresponding spectral radius $\rho(Q_{\omega_{opt}})$.
- (b) Consider the $n \times n$ matrix A defined by its entries

$$a_{ij} = \begin{cases} 4, & i = j, \\ -1, & |i - j| = 1, \\ 0, & \text{otherwise} \end{cases}$$

The eigenvalues of A are $\lambda_i = 4 - 2\cos(\pi ih)$, $i = 1, \dots, n$, and $h = \frac{1}{n+1}$. Define the Jacobi iteration matrix, Q_J , compute its eigenvalues and the spectral radius, $\rho(Q_J)$. Then, for the optimal SOR iteration applied to a problem with A and using the result from (a) compute the optimal value of the parameter, ω and the corresponding spectral radius $\rho(Q_{\omega_{opt}})$ (what is its value as $h \rightarrow 0$?)

- (c) Using the results from (b), compute the dependence between the rate of convergence, $R_{\infty}(Q_{\omega_{opt}})$ and n , as well as between the number of iterations, N and n .

Problem 2. (ODEs)

- (a) Give a definition of a linear multistep method (LMM) for solving $y'(t) = f(t, y)$.
- (b) Formulate the necessary and sufficient conditions for the convergence of a LMM.
- (c) Check whether or not the following LMM is convergent:

$$y_{n+4} - y_n = \frac{h}{3}(8f_{n+3} - 4f_{n+2} + 8f_{n+1}).$$

Problem 3. (Orthogonal polynomials) The Legendre equation is given by

$$(1 - x^2)y''(x) - 2xy'(x) + n(n + 1)y(x) = 0, \quad x \in (-1, 1). \quad (1)$$

- (a) Show that for every integer n , equation (1) has a polynomial solution of order n .
- (b) Let $p_n, n = 0, 1, 2, \dots$ be polynomial solutions of (1). Show that these polynomials are orthogonal with respect to the $L^2(-1, 1)$ inner product. These polynomials are also called Legendre polynomials.
- (c) Show that Legendre polynomial p_n has exactly n distinct roots over the interval $(-1, 1)$.

Problem 4. (Nonlinear equations) Consider the following polynomial equations in the complex domain

$$z^4 = 1. \quad (2)$$

- (a) Rewrite (2) as system of nonlinear equations in \mathcal{R}^2 .
- (b) Write the appropriate formulas of Newton's method for a system of nonlinear equations. Compute all terms used in these formulas for (2).
- (c) Starting from $z_0 = 1 + i$ compute two iterates of z_1, z_2 Newton's method. Determine whether or not Newton method converges in this case.

Problem 5. (Nonlinear equations) Let p be a monic (leading coefficient equal to 1) polynomial of degree n with n distinct real roots r_1, r_2, \dots, r_n satisfying $r_1 < r_2 < \dots < r_n$. Let $\{x_k\}_{k=0}^{\infty}$ be the sequence of iterates generated by Newton's method applied to the equation $p(x) = 0$.

- (a) Use Rolle's Theorem to show that $p'(x) > 0$ if $x \geq r_n$.
- (b) Use part (a) to help you argue that if $x_0 > r_n$, then $r_n < \dots < x_{k+1} < x_k < \dots < x_0$, for $k = 1, 2, \dots$.

Problem 6. (ODE, Linear System) Consider the m -dimensional linear initial value problem given by

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t), \quad \vec{y}(0) = \vec{y}_0, \quad 0 \leq t \leq T. \quad (3)$$

For $n = 1, 2, \dots$. Let $t_k^n = kh$, $h = T/n$, $k = 0, 1, 2, \dots, n$ and let \vec{y}_k^n denote an approximation to $\vec{y}(t_k^n)$.

- (a) Show that the use of the trapezoid rule to derive an implicit numerical approximation scheme for (3) results in

$$\left(I - \frac{h}{2}A\right) \vec{y}_{k+1}^n = \left(I + \frac{h}{2}A\right) \vec{y}_k^n.$$

- (b) Consider the $l + 1$ dimensional linear system of form (3) given by

$$\begin{aligned} u_0'(t) &= \frac{u_1(t) - 2u_0(t)}{r^2}, \\ u_j'(t) &= \frac{u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)}{r^2}, \quad j = 1, \dots, l-1, \\ u_l'(t) &= \frac{-2u_l(t) + u_{l-1}(t)}{r^2}, \end{aligned}$$

where $r > 0$. Show that for this system, the matrix $(I - hA/2)$ is nonsingular.

Problem 7. (Least Square Minimization) Given data $\{(y_i, x_{1,i}, x_{2,i}, \dots, x_{m-1,i})\}_{i=1}^n$, consider the least squares problem of fitting a linear model of the form

$$y = \beta_0 + \beta_1 x_1 + \dots + \beta_{m-1} x_{m-1}$$

which minimizes the sum of the squares of the residuals given by

$$J(\vec{\beta}) = \sum_{i=1}^n \left(y_i - \sum_{j=0}^{m-1} \beta_j x_{j,i}\right)^2,$$

with $x_{0,i} = 1$, $i = 1, 2, \dots, n$, and $\vec{\beta} = [\beta_0, \dots, \beta_{m-1}]^T$.

- (a) Show that J is minimized by choosing $\vec{\beta}^*$ is a solution to the linear system

$$X^T X \vec{\beta}^* = X^T \vec{y},$$

where $\vec{y} = [y_1, \dots, y_n]^T$ and X is the $n \times m$ matrix given by $[X]_{i,j} = x_{j,i}$, $i = 1, \dots, n$, $j = 0, \dots, m - 1$.

- (b) Show that the linear system given in part (a) has a unique solution if and only if $\text{rank}(X) = m$.
- (c) Suppose that $y_1 = \beta + \epsilon_1$ and $y_2 = 2\beta + \epsilon_2$ where ϵ_1 and ϵ_2 are measurement errors. Use the least square method outlined above to find an expression for the least square estimate β^* for β in term of y_1 and y_2 .

Numerical Analysis Qualifying Exam May 2001

Part I

1. Computational Linear Algebra

- (a) Write the equations -or a pseudo-code program - for the LU decomposition of a real matrix $A = (a_{ij})$ without pivoting.
- (b) Write the equations (or pseudo-code) for forward and back substitution to solve $Ax = b$, given $A = LU$.
- (c) Define the condition number of A . Use it to specify a priori and a posteriori estimates of rounding error in solving $Ax = b$.
- (d) Write the equations specifying the Gauss-Seidel iteration to solve $Ax = b$. Give a sufficient condition for its convergence.

2. Solution of Nonlinear Equations

- (a) State and prove the contractive mapping theorem for the iteration
* $x_{n+1} = g(x_n)$
to find the fixed point of the mapping $g : V \rightarrow V$, where V is an appropriate vector space.
- (b) State Newton's method to solve $f(x) = 0$. Express it as an iteration of the form * above.
- (c) Prove quadratic convergence of Newton's method in the 1-dimensional case.
- (d) Formulate Newton's method to solve $x^2 - a = 0$.

3. Polynomial Approximation

Let $f(x)$ be a real continuous function on $[0, 1]$.

- (a) Define the Best Approximating Polynomial (BAP) for f of degree $\leq n$. Prove that it exists.
- (b) Specify the Remes Single Exchange Algorithm to compute the BAP of degree $\leq n$ for f restricted to a finite set of N points in $[0, 1]$.
- (c) In part 3(b), discuss what happens as $N \rightarrow \infty$.
- (d) Find the BAP of degree 0 for $f(x) = x^2$ on $[0, 1]$. Verify the equi-oscillation condition. Use this condition to calculate the BAP of degree 1.

4. Polynomial Interpolation

- (a) Give the Lagrange form of the interpolating polynomial $L_n(x)$ of degree $\leq n$ for values of a function $f(x)$ on a set of $n + 1$ points.
- (b) Give the Newton divided difference form of $L_n(x)$.
- (c) Derive the formula for the error $f(x) - L_n(x)$ for x in an interval $[a, b]$ and assuming sufficient differentiability of f .
- (d) Does $L_n(x)$ always converge to $f(x)$ as $n \rightarrow \infty$? Justify your answer.

Numerical Analysis Qualifying Exam May 2001

Part 2

5. Ordinary Differential Equations

Given the initial value problem

$$(IVP) \quad dx/dt = f(t, x), \quad x(t_0) = a.$$

- Discuss predictor-corrector Adams-Moulton (AM) finite difference methods to solve IVP. Give an example of one such method. (If you don't recall the coefficient values, just use symbols for the coefficients.) What is the local one-step error for step-size h ? What is the cumulative error?
- Discuss Runge-Kutta (RK) methods to solve IVP. As an example, give the formulas for the fourth-order RK method. (If you don't recall coefficient values, use symbols.) How would you use this method in conjunction with the fourth-order AM method?
- State the theorem on convergence and stability of linear multistep difference methods for solving IVP.

6. Ordinary Differential Equations

In IVP above, let $f(t, x) = -\lambda x$, $\lambda > 0$, x a real variable.

- Apply the Euler method to this case. Derive the step-size h needed for absolute stability.
- Apply the backward Euler method and find the step-size for absolute stability.
- Apply the trapezoidal rule and find h for absolute stability.
- Define the BDF methods for solving IVP. Give an example of a BDF method.

7. Least Squares Approximation

Let A be an $m \times n$ real matrix and b an m -dimensional real vector. Consider the equation (*) $Ax = b$.

- State a necessary and sufficient condition that (*) have a solution. Express this in terms of rank (A, b) , where (A, b) is the augmented matrix. Discuss uniqueness of the solution.
- Define the least squares solution x^* of (*). Prove that x^* always exists. Discuss uniqueness of x^* . What methods yield the x^* of minimum l_2 norm?
- Define the pseudo-inverse of A . Outline a method to compute it.
- Let $f(x)$ be defined on $[0, 1]$. Define the least squares polynomial p of degree $\leq n$ for f on $[0, 1]$.
- Let E be a finite subset of points in $[0, 1]$. Derive the normal equations to compute p on the set E . Discuss the condition number of the normal matrix as a function of n . How large can n be in practice?

Numerical Analysis Screening Exam - Feb. 2002

Problem 1. Linear Systems

- Find matrices A for which the number of solutions to $Ax = b$ is
 - 0 or 1, depending on b ,
 - 1 or ∞ , depending on b ,
 - 0 or ∞ , depending on b ,
 - 1, regardless of b .
- You want to compute $y = AB^{-1}Cx$, where $x, y \in R^n$, $A, B, C \in R^{n \times n}$, and B is tridiagonal. Organize computations with minimal computational cost (count only the highest order terms).
- Consider the following linear system

$$Ax = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & & \\ 1 & & 3 & \\ 1 & & & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 4 \\ 5 \end{pmatrix} = b.$$

Explain the fill-in phenomenon in Gaussian elimination on this example. Suggest a remedy and show that it works.

Problem 2. Linear Least Squares

Let A be a given $m \times n$ real matrix, b a given real m -dimensional vector and x an unknown n -dimensional vector.

- State a necessary and sufficient condition for the equation $Ax = b$ to have a solution.
- Define what is meant by a least solution x^* of $Ax = b$. Prove that x^* always exists.
- Discuss the question of the uniqueness of x^* in terms of the range and null space of A .

If there are two least squares solutions x^* and y^* , what can be said about $x^* - y^*$? Prove your answer.

- Define the pseudo-inverse A^\dagger of A . Define the singular values and singular vectors of A . State the result on the singular values decomposition (SVD) of A and its relation to A^\dagger .
- Show how to compute $A^\dagger b$ using the SVD.

Numerical Analysis Screening Exam - Sept. 2002

Problem 2. Linear Least Squares

Let A be a given $m \times n$ real matrix, b a given real m -dimensional vector and x an unknown n -dimensional vector.

- a. Define a least squares solution (LSS) to (*) $Ax = b$.
- b. Does a least squares solution to (*) always exist? If the solution exists, is it unique?
- c. Define the Normal Equations and two other methods to compute the LSS. Why (and when? - give an example) the latter are preferable to the Normal Equation method?

Numerical Analysis Screening Exam Feb. 11, 2003

Problem 1. Solution of Linear Equations by Direct Methods

Given a non-singular $n \times n$ real matrix $A = (a_{ij})$.

(a) Write the equations (or a pseudo-code program) for an algorithm to compute the LU decomposition of A without pivoting. (Assume that the LU factors exist.) How many arithmetic operations are required for a general matrix?

(b) Write the equations (or pseudo-code) for forward and back substitution to solve $Ax = b$, given that $A = LU$. How many operations are required?

(c) Now do not assume that A has an LU decomposition. Prove that there is a permutation matrix, P , such that PA has an LU decomposition. Describe how P can be found by a partial pivoting procedure included in the algorithm in part (a).

(d) Define the condition number, γ , of A . Use γ to specify a formula for an *a priori* estimate of rounding error in solving the system $Ax = b$. Define what is meant by an ill-conditioned system. What value of γ would suggest ill-conditioning? What can be done in the computer to offset the effects of ill-conditioning?

Numerical Analysis Screening Exam Feb. 11, 2003

Problem 2. Iterative Solution of Linear Equations

Let A be a real non-singular $n \times n$ matrix and b a real n -dimensional vector. Consider the equation

$$(LE) \quad Ax = b.$$

Let $A = M - N$ be a splitting of A such that M is non-singular.

(a) Use the splitting to write the equation for an iteration that computes a sequence of vectors, $x^{(m)}$. If the sequence converges, prove that the limit is the solution of (LE).

(b) Define $Q = M^{-1}N$. Give a sufficient condition on Q for the convergence of $x^{(m)}$.

Prove convergence using this condition. What is the rate of convergence? Explain your answer.

(c) Give the splitting for the Jacobi iteration. Give a property of A in terms of its diagonal elements that implies the sufficient condition in (b) holds for the Jacobi iteration. Prove this result.

(d) Give the splitting for the Gauss-Seidel iteration and prove a property of A that is sufficient for convergence.

Numerical Analysis Screening Exam Feb. 11, 2003

Problem 3. Matrix Eigenvalues and Eigenvectors

Let A be an $n \times n$ real matrix.

- (a) Define the eigenvalues and eigenvectors of A . How many eigenvalues are there? Prove your answer. Are all eigenvalues necessarily real? Explain your answer. Now suppose A is symmetric, so that $A = A^T$. What can be said about the eigenvalues and eigenvectors? Prove your answer.
- (b) Let $A = A^T$. Using 2D rotations, describe the Jacobi method to compute the eigenvalues and eigenvectors of A . Give 2 possible algorithms for applying the rotations. What is the rate of convergence to the eigenvalues?
- (c) Give the main equations for the QR algorithm to compute the eigenvalues of A by computing a sequence of matrices A_k . Prove that the A_k have the same eigenvalues as A . Now assume that sequence A_k converges to U . What is the structure of U if A has all real eigenvalues? What is the structure of U if A has some complex eigenvalues? Describe the shifted QR algorithm.

Problem 4. Linear Least Squares

Let A be a given $m \times n$ real matrix, b a given m -dimensional vector and x an unknown n -dimensional vector. Consider the equation .

$$(LS) \quad Ax = b .$$

(a) State a necessary and sufficient condition on the rank of the augmented matrix (A, b) for LS to have a solution. Construct a simple case for $m = 3, n = 2$ where $\text{rank}(A) = 2$ and there is no solution.

(b) Letting $f(x) = \|Ax - b\|_2^2$, define a *least squares solution*, x^* , of (LS). Write the normal equations that correspond to equation LS. Show that $f(x) = \langle A^T A x, x \rangle - 2 \langle A^T b, x \rangle + \langle b, b \rangle$ where $\langle \cdot, \cdot \rangle$ denotes inner-product. Compute the gradient $\nabla f(x)$ and prove that x^* satisfies the normal equations. When is x^* unique ?

(d) Define the pseudo-inverse, A^\dagger , of A . Define the singular values and singular vectors of A . State the result on the singular values decomposition (SVD) of A and its relation to A^\dagger . Outline the steps of an algorithm to compute the SVD.

Page 1 of Numerical Analysis Screening Exam Sept. 22, 2003

Do all 3 problems. Each problem counts 33 and 1/3 points and each part is equally weighted.

Problem 1. Iterative Solution of Linear Equations

Let A be a real non-singular $n \times n$ matrix and b a real n -dimensional vector. Consider the equation (1) in problem 1. Let $A = M - N$ be a splitting of A such that M is non-singular.

(a) Use the splitting to write the equation for an iteration that computes a sequence of vectors, $x^{(m)}$. If the sequence converges, prove that the limit is the solution of (1).

(b) Define $Q = M^{-1}N$. Give a sufficient condition on Q for the convergence of $x^{(m)}$. Prove convergence using this condition. What is the rate of convergence?

(c) Give the splitting for the Jacobi iteration. State a property of A in terms of its diagonal elements that implies the sufficient condition in (b) holds for the Jacobi iteration. Prove this result. (Hint: Define diagonal dominant matrix)

(d) Give the splitting for the Gauss-Seidel iteration and prove a property of A that is sufficient for convergence. Define a family of relaxation iterations with parameter T which yields the Gauss-Seidel iteration when $T = 1$. Discuss how to choose T to minimize the spectral radius of Q_T .

Problem 2. Matrix Eigenvalues and Eigenvectors

Let A be an $n \times n$ real matrix.

(a) Define the eigenvalues and eigenvectors of A . How many eigenvalues are there? Prove your answer. Are all eigenvalues necessarily real? Explain your answer. Now suppose A is symmetric, so that $A = A^T$. What can be said about the eigenvalues and eigenvectors? Prove your answer.

(b) Let $A = A^T$. Using 2D rotations, describe the Jacobi method to compute the eigenvalues and eigenvectors of A . Give 2 possible algorithms for applying the rotations to annihilate elements. What is the rate of convergence to the eigenvalues?

(c) Give the main equations for the QR algorithm to compute the eigenvalues of a general matrix A by computing a sequence of matrices A_k . Prove that the A_k have the same eigenvalues as A . Now assume that sequence A_k converges to matrix U . What is the structure of U if A has all real eigenvalues? What is the structure of U if A has some complex eigenvalues?

(d) Describe the shifted QR algorithm. How does it improve convergence?

Problem 3. Linear Least Squares

Let A be an $m \times n$ real matrix, b an m -dimensional vector and x an unknown n -dimensional vector. Consider the equation

$$(LS) \quad Ax = b.$$

(a) Define the range of A and the rank of A . State a necessary and sufficient condition on the rank of the augmented matrix (A, b) for (LS) to have a solution. Express this in terms of b and the range of A . Use this condition to construct a simple case for $m = 3$, $n = 2$ where $\text{rank}(A) = 2$ and there is no solution.

(b) Let $f(x) = \|Ax - b\|_2^2$. Define a *least squares solution*, x^* , of (LS) in terms of $f(x)$ and prove that x^* always exists. Write the normal equations that correspond to equation (LS). Derive the formula

$$f(x) = \langle A^T A x, x \rangle - 2 \langle A^T b, x \rangle + \langle b, b \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes inner-product. Compute the gradient $\nabla f(x)$ and use it to prove that x^* satisfies the normal equations. When is x^* unique?

(d) Define the pseudo-inverse, A^\dagger , of A . Show that $x^* = A^\dagger b$ is a least squares solution of (LS). What can be said about the norm of x^* ?

(e) Define the singular values and singular vectors of A . State the result on the singular values decomposition (SVD) of A and its relation to A^\dagger . Outline the steps of an algorithm to compute the (SVD).

Do all 3 problems. Each problem counts 33 and 1/3 points and each part is equally weighted.

Problem 1. Iterative Solution of Linear Equations

Let A be a real non-singular $n \times n$ matrix and b a real n -dimensional vector. Consider the equation (1) in problem 1. Let $A = M - N$ be a splitting of A such that M is non-singular.

(a) Use the splitting to write the equation for an iteration that computes a sequence of vectors, $x^{(m)}$. If the sequence converges, prove that the limit is the solution of (1).

(b) Define $Q = M^{-1}N$. Give a sufficient condition on Q for the convergence of $x^{(m)}$. Prove convergence using this condition. What is the rate of convergence?

(c) Give the splitting for the Jacobi iteration. State a property of A in terms of its diagonal elements that implies the sufficient condition in (b) holds for the Jacobi iteration. Prove this result. (Hint: Define diagonal dominant matrix)

(d) Give the splitting for the Gauss-Seidel iteration and prove a property of A that is sufficient for convergence. Define a family of relaxation iterations with parameter T which yields the Gauss-Seidel iteration when $T = 1$. Discuss how to choose T to minimize the spectral radius of Q_T .

Problem 2. Matrix Eigenvalues and Eigenvectors

Let A be an $n \times n$ real matrix.

(a) Define the eigenvalues and eigenvectors of A . How many eigenvalues are there? Prove your answer. Are all eigenvalues necessarily real? Explain your answer. Now suppose A is symmetric, so that $A = A^T$. What can be said about the eigenvalues and eigenvectors? Prove your answer.

(b) Let $A = A^T$. Using 2D rotations, describe the Jacobi method to compute the eigenvalues and eigenvectors of A . Give 2 possible algorithms for applying the rotations to annihilate elements. What is the rate of convergence to the eigenvalues?

(c) Give the main equations for the QR algorithm to compute the eigenvalues of a general matrix A by computing a sequence of matrices A_k . Prove that the A_k have the same eigenvalues as A . Now assume that sequence A_k converges to matrix U . What is the structure of U if A has all real eigenvalues? What is the structure of U if A has some complex eigenvalues?

(d) Describe the shifted QR algorithm. How does it improve convergence?

Problem 3. Linear Least Squares

Let A be an $m \times n$ real matrix, b an m -dimensional vector and x an unknown n -dimensional vector. Consider the equation

$$(LS) \quad Ax = b.$$

(a) Define the range of A and the rank of A . State a necessary and sufficient condition on the rank of the augmented matrix (A, b) for (LS) to have a solution. Express this in terms of b and the range of A . Use this condition to construct a simple case for $m = 3$, $n = 2$ where $\text{rank}(A) = 2$ and there is no solution.

(b) Let $f(x) = \|Ax - b\|_2^2$. Define a *least squares solution*, x^* , of (LS) in terms of $f(x)$ and prove that x^* always exists. Write the normal equations that correspond to equation (LS). Derive the formula

$$f(x) = \langle A^T A x, x \rangle - 2 \langle A^T b, x \rangle + \langle b, b \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes inner-product. Compute the gradient $\nabla f(x)$ and use it to prove that x^* satisfies the normal equations. When is x^* unique?

(d) Define the pseudo-inverse, A^\dagger , of A . Show that $x^* = A^\dagger b$ is a least squares solution of (LS). What can be said about the norm of x^* ?

(e) Define the singular values and singular vectors of A . State the result on the singular values decomposition (SVD) of A and its relation to A^\dagger . Outline the steps of an algorithm to compute the (SVD).

Numerical Analysis Screening/Qualifying Exam

January 26, 2004

USC Mathematics Department

Instructions to students: There are 4 problems. Each counts 25 points. Please write your name on the exam paper.

Numerical Analysis Screening Exam - Jan. 2004

Problem 1. Linear Systems

Consider the equation $Ax = b$, where A is an $n \times n$ real matrix and b an n -dimensional real vector.

- a. What is the LU-decomposition of A and how is it related to Gaussian Elimination? Does it always exist? If not, give sufficient conditions for its existence.
- b. Show that if the k th diagonal entry of an upper triangular matrix is zero, then its first k columns are linearly dependent.
- c. What is meant by partial pivoting in Gaussian elimination? Mention two classes of matrices for which Gaussian elimination can be performed stably without pivoting.
- d. What is the Cholesky factorization for a symmetric positive definite matrix? What is its relation to Gaussian elimination? Give an example of a symmetric matrix for which Cholesky factorization does not exist.
- e. Show that if A is symmetric positive definite so is its inverse A^{-1} .

Problem 2 Iterative methods

Consider the equation $Ax = b$, where A is an $n \times n$ real matrix and b an n -dimensional real vector.

a. Name and define three iterative methods that can be derived from the splitting $A = M - N$.

b. Define the average and asymptotic rate of convergence for an iterative method $x^{k+1} = Bx^k + c$. If the spectral radius of B is $\rho(B) = 1 - \frac{1}{n}$, estimate how many iterations are needed to reduce the initial error by 10^6 times.

c. Let the $n \times n$ symmetric positive definite matrix A have the condition number $\kappa(A) = 100$. Estimate roughly how many flops are required to solve $Ax = b$ by conjugate gradient iteration to ten-digit accuracy. Compare this estimate with the flop cost for Cholesky factorization.

Numerical Analysis Exam Jan. 26, 2004

3. (Least squares) Let A be a linear operator mapping a real n -dimensional vector space, V^n , into a real m -dimensional vector space V^m . Both spaces have an inner-product $\langle u, v \rangle$. The *adjoint* of A is a linear operator A^* mapping V^m into V^n and satisfying the relation $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x \in V^n$ and $y \in V^m$.

(a) Prove that $A^{**} = A$.

(b) Prove that if bases are chosen and A is represented by an $m \times n$ matrix relative to these bases, then A^* is represented by the transpose of the matrix.

(c). The *null space* $N_A = \{x : Ax = 0\}$. The *range* $R_A = \{Ax : x \in V^n\}$. Let $r(A) = \text{dimension of } R_A$ and $v(A) = \text{dimension of } N_A$. Prove that $r(A) + v(A) = n$. The *orthogonal complement* of N_A is the subspace $N_A^\perp = \{x : \langle x, u \rangle = 0 \text{ for all } u \in N_A\}$. Prove that A maps N_A^\perp onto R_A in a 1:1 way, so that $\dim(N_A^\perp) = r(A)$ and for any $x \in V^n$, $x = x_{N(A)} + x_{N(A)^\perp}$.

(d) Let $f(x) = \|Ax - b\|^2$, where $x \in V^n$ and $b \in V^m$. The norm is the l_2 norm. Using algebraic properties of the inner-product, derive the following formulas:

(i) $f(x) = \langle A^*Ax, x \rangle - 2\langle A^*b, x \rangle + \|b\|^2$,

(ii) $f(x + h) = f(x) + 2\langle A^*Ax - A^*b, h \rangle + \langle A^*Ah, h \rangle$.

x^- is called a *global minimum* of f if $f(x^-) \leq f(x)$ for all x . Prove that f has a global minimum. (Hint: Let $R(A) = R_A$. To minimize $f(x)$, find the vector in $R(A)$ which is closest to b . Note that $b = b_{R(A)} + b_{R(A)^\perp}$ and $\exists x^-$ such that $Ax^- = b_{R(A)}$.)

Use (ii) to prove that a necessary and sufficient condition on x^- is $A^*Ax^- = A^*b$ (normal equations). Does A^*A always have an inverse? How is its rank related to $r(A)$?

(e) Define the *pseudoinverse*, A^\dagger , of A . It can be proved that $AA^\dagger b = b_{R(A)}$. What does this imply about $A^\dagger b$? State the Singular Values Decomposition (SVD) Theorem for computing A^\dagger .

4 (Eigenvalue Problems) The preliminary reduction to upper Hessenberg form would be of little use if the steps of the QR algorithm did not preserve the structure. Fortunately they do.

- a) In the QR factorization $A = QR$ of an upper Hessenberg matrix A , which entries of R are in general nonzero? Which entries of Q ?
- b) Show that the upper Hessenberg structure is recovered when the product RQ is formed.
- c) Explain how the Givens rotation or 2×2 Householder reflections can be used in the computation of the QR factorization of a tridiagonal matrix, reducing the operation count.

Screening Exam in Numerical Analysis, Fall 2004

I. Linear systems

1. Show that the 2-norm is invariant under orthogonal transformations.
2. We want to compute $y = AB^{-1}Cx$, where $x, y \in \mathbb{R}^n$, $A, B, C \in \mathbb{R}^{n \times n}$, and B is tridiagonal. Organize computations with minimal computational cost (count only the highest order terms). What is the total cost in flops.
3. It is known that the components of the exact solution to a linear system $Ax = b$ range from 10^{-1} to 10^3 and that condition number of A is about 10^4 . What must the machine precision be in order to ensure that the smallest component of $x = A \setminus b$ (in Matlab) has at least five significant digits of accuracy? No proof is necessary, just a reasonable heuristic argument.

4. Let $A = \begin{bmatrix} .780 & .563 \\ .913 & .659 \end{bmatrix}$, $b = \begin{bmatrix} .217 \\ .254 \end{bmatrix}$.

On two different computers with about the same machine precision two solutions were computed to the system $Ax = b$:

$$x_1 = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}, \text{ and } x_2 = \begin{bmatrix} .341 \\ -.087 \end{bmatrix}.$$

- 4a. Compute the residual vectors corresponding to these two solutions.
- 4b. Compute the error vectors corresponding to these two solutions.
(Note: b is equal to the difference between the columns of A).
- 4c. How large are norms of the residual and error vectors? Comment on the relations between them and give explanations to your observations.

II. Iterative solutions.

1. Consider solving a system $Ax = b$, where $x, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ using conjugate gradient (cg) iterations.
 - 1a. What are the assumptions on A ?
 - 1b. Define the properties of residuals r_k and search directions p_k (without actually writing the algorithm).
 - 1c. Give an estimate of rate of convergence (in which norm?) if A satisfies the assumptions in 1a.
 - 1d. When is cg solver advantageous over a direct solver?
2. Define a preconditioned conjugate gradient (pcg) iterations for solving $Ax = b$.
 - 2a. What properties are required from the preconditioner M ?
 - 2b. Give an example of a preconditioner for a given A and justify your choice.

III Let A be an $n \times n$ real matrix.

1. Define the eigenvalues and eigenvectors of A as solutions of matrix-vector equation. Show that the eigenvalues are solutions of an n -th degree real polynomial equation. How many eigenvalues are there? Prove your answer. Are all eigenvalues necessarily real. Explain your answer.
2. Give the main equations for the QR algorithm to compute the eigenvalues of A by computing a sequence of matrices A_k . Prove that A_k have the same eigenvalues as A . Now assume that the sequence A_k converges to U . What is the structure of U if A has all real eigenvalues? What is the structure of U if A has some complex eigenvalues? Describe the shifted QR algorithm.
3. Let $A \in R^{m \times m}$ be an arbitrary matrix. The set of all Rayleigh quotients of A , corresponding to all nonzero vectors $x \in R^m$, is known as the field of values or numerical range of A , a subset of the complex plane denoted by $W(A)$

$$W(A) = \left\{ r \in R; r = \frac{x^T A x}{x^T x}, x \neq 0 \right\}.$$

The convex hull of all the eigenvalues $\{\lambda_i, i = 1, \dots, m\}$ of A is defined by

$$K(A) = \left\{ w \in R; w = \sum_{i=1}^m a_i \lambda_i, 0 \leq a_i \leq 1, \sum_{i=1}^m a_i = 1, i = 1, \dots, m \right\}.$$

Show that if A is normal (i.e., $AA^* = A^*A$), then $K(A) = W(A)$. (Hint: Prove both inclusions).

Screening Exam Sept. 2004

(IV) Least Squares Problem. A is an $m \times n$ real matrix mapping \mathbb{R}^n into \mathbb{R}^m . $\langle x, u \rangle$ denotes inner-product. All norms are l_2 norms. The null space $N_A = N(A) = \{x : Ax = 0\}$. The range $R_A = R(A) = \{Ax : x \in \mathbb{R}^n\}$. $\text{rank } r(A) = \text{dimension of } R_A$. The orthogonal complement of $N(A)$ is the subspace $N_A^\perp = \{x : \langle x, u \rangle = 0 \text{ for all } u \in N_A\}$.

(1) Prove: A maps N_A^\perp onto R_A in a 1:1 way, so that $\dim(N_A^\perp) = r(A)$ and $\mathbb{R}^n = N_A \oplus N_A^\perp$.

(2) Let $f(x) = \|Ax - b\|^2$, where $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

(a) Using algebraic properties of the inner-product, derive the following formulas:

(i) $f(x) = \langle A^*Ax, x \rangle - 2\langle A^*b, x \rangle + \|b\|^2$,

(ii) $f(x+h) = f(x) + 2\langle A^*Ax - A^*b, h \rangle + \langle A^*Ah, h \rangle$.

(b) Prove that f has a global minimum. (Hint: To minimize $f(x)$ first find the vector in $R(A)$ which is closest to b . Since $b = b_{R(A)} + b_{R(A)^\perp}$, the closest vector is $b_{R(A)}$. Prove this and there exists \tilde{x} such that $A\tilde{x} = b_{R(A)}$.)

(c) Prove that a necessary and sufficient condition on \tilde{x} is $A^*A\tilde{x} = A^*b$.

(Hint: Use (ii).)

(3) The *pseudoinverse* of A is a linear operator A^\dagger on \mathbb{R}^m to \mathbb{R}^n satisfying the Moore-Penrose conditions:

(Gen) $AA^\dagger A = A$ and $A^\dagger A A^\dagger = A^\dagger$;

(Sym) $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$.

Assume A^\dagger exists. Prove that $A^{\dagger\dagger} = A$.

(4) It can be proved that $AA^\dagger = P_{R(A)}$ and $A^\dagger A = P_{N(A)^\perp}$, where P denotes the projection onto the subspace.

(a) Use this to prove the following:

Theorem. $A^\dagger b$ is a solution of the least squares problem, that is,

$$\|AA^\dagger b - b\| \leq \|Ax - b\| \text{ for all } x \in \mathbb{R}^n.$$

(Hint: $AA^\dagger b = b_{R(A)}$, where $b = b_{R(A)} + b_{R(A)^\perp}$. For any x ,

$$\|Ax - b\|^2 = \|Ax - b_{R(A)}\|^2 + \|b_{R(A)^\perp}\|^2 \geq \|b_{R(A)^\perp}\|^2. \text{ Explain and justify each step.}$$

(b) State the Singular Values Decomposition (SVD) Theorem for computing A^\dagger .

Numerical Analysis Exam Jan. 26, 2004

3. (Least squares) Let A be a linear operator mapping a real n -dimensional vector space, V^n , into a real m -dimensional vector space V^m . Both spaces have an inner-product $\langle u, v \rangle$. The *adjoint* of A is a linear operator A^* mapping V^m into V^n and satisfying the relation $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x \in V^n$ and $y \in V^m$.

(a) Prove that $A^{**} = A$.

(b) Prove that if bases are chosen and A is represented by an $m \times n$ matrix relative to these bases, then A^* is represented by the transpose of the matrix.

(c). The *null space* $N_A = \{x : Ax = 0\}$. The *range* $R_A = \{Ax : x \in V^n\}$. Let $r(A) = \text{dimension of } R_A$ and $v(A) = \text{dimension of } N_A$. Prove that $r(A) + v(A) = n$. The *orthogonal complement* of N_A is the subspace $N_A^\perp = \{x : \langle x, u \rangle = 0 \text{ for all } u \in N_A\}$. Prove that A maps N_A^\perp onto R_A in a 1:1 way, so that $\dim(N_A^\perp) = r(A)$ and for any $x \in V^n$, $x = x_{N(A)} + x_{N(A)^\perp}$.

(d) Let $f(x) = \|Ax - b\|^2$, where $x \in V^n$ and $b \in V^m$. The norm is the l_2 norm. Using algebraic properties of the inner-product, derive the following formulas:

(i) $f(x) = \langle A^*Ax, x \rangle - 2\langle A^*b, x \rangle + \|b\|^2$,

(ii) $f(x+h) = f(x) + 2\langle A^*Ax - A^*b, h \rangle + \langle A^*Ah, h \rangle$.

x^- is called a *global minimum* of f if $f(x^-) \leq f(x)$ for all x . Prove that f has a global minimum. (Hint: Let $R(A) = R_A$. To minimize $f(x)$, find the vector in $R(A)$ which is closest to b . Note that $b = b_{R(A)} + b_{R(A)^\perp}$ and $\exists x^-$ such that $Ax^- = b_{R(A)}$.)

Use (ii) to prove that a necessary and sufficient condition on x^- is $A^*Ax^- = A^*b$ (normal equations). Does A^*A always have an inverse? How is its rank related to $r(A)$?

(e) Define the *pseudoinverse*, A^\dagger , of A . It can be proved that $AA^\dagger b = b_{R(A)}$. What does this imply about $A^\dagger b$? State the Singular Values Decomposition (SVD) Theorem for computing A^\dagger .

4 (Eigenvalue Problems) The preliminary reduction to upper Hessenberg form would be of little use if the steps of the QR algorithm did not preserve the structure. Fortunately they do.

- a) In the QR factorization $A = QR$ of an upper Hessenberg matrix A , which entries of R are in general nonzero? Which entries of Q ?
- b) Show that the upper Hessenberg structure is recovered when the product RQ is formed.
- c) Explain how the Givens rotation or 2×2 Householder reflections can be used in the computation of the QR factorization of a tri-diagonal matrix, reducing the operation count.

Numerical Analysis Screening Exam Spring 2005

(Do all 4 problems. Each counts 25 points.)

1. Linear Equations, Direct Methods of Solution of $Ax = b$.

Let A be an $n \times n$ real non-singular matrix.

(a) Describe the Gauss elimination algorithm without pivoting. What properties should A have to apply this method? Describe the Choleski factorization method to solve $Ax = b$. What properties should A have to apply this method?

(b) Compare the computational costs of the two methods in (a).

(c) Give an example of a matrix A for which Gauss elimination without pivoting is unstable. What does "unstable" mean?

(d) Explain a pivoting procedure for Gauss elimination. How does it help to reduce rounding errors?

Numerical Analysis Screening Exam Spring 2005

2. Eigenvalue Problems

Let A be a real $n \times n$ matrix.

(a) Define a *diagonalization* of A , a *unitary diagonalization* of A and a *Schur factorization* of A . Specify conditions on A for each of them to exist.

(b) Give an argument to show that an algorithm to compute eigenvalues of an arbitrary matrix A must be an iterative one.

(c) For a general method to compute eigenvalues explain the two phases: one involving Hessenberg matrices and the other elementary orthogonal matrices (also known as Householder reflections). What kind of matrix is obtained from phases 1 and 2 for a general A and for a symmetric A ?

(d) Describe the QR algorithm with shifts. When it converges, what is its order of convergence? Give the conditions on the eigenvalues for this order to be achieved.

Numerical Analysis Screening Exam Spring 2005

3. Iterative Solution of $Ax = b$

A is an $n \times n$ real non-singular matrix, $b \in \mathbb{R}^n$.

- (a) Define what is meant by a splitting of A into two matrices M and N , where M is non-singular.
- (b) With respect to a splitting, derive an iterative method to solve $Ax = b$. State and prove a sufficient condition for convergence in terms of M and N .
- (c) Give the splitting for the Jacobi iteration. State and prove an easily verifiable sufficient condition on A for convergence. Write the iteration formula in component form. Using vector norms, give a test for stopping the iteration.
- (d) Give the splitting for the Gauss-Seidel (G-S) iteration. State a sufficient condition on A for convergence of G-S. Write the formula for the G-S iteration in component form. Give a stopping criterion based on components of successive iterates.
- (e) Define the spectral radius $r_\sigma(Q)$ of an $n \times n$ matrix Q . Let an iteration be defined by $x_{k+1} = Qx_k + c$. Define the asymptotic rate of convergence $R(Q)$ of the iterative method in terms of $r_\sigma(Q)$. Give a lower bound of $R(Q)$ in terms of some matrix norm $\|Q\|$. Explain how to use $R(Q)$ to estimate the number of iterations needed to reduce the initial error by 10^{-m} .

Screening Exam in Numerical Analysis, Spring 2005

4. Least Squares

1. Consider a linear system $Ax = b$ (1),
where A is a rectangular matrix $A \in R^{m \times n}$, $m > n$, $x \in R^n$, $b \in R^m$.
 - a. Give condition(s) under which a solution to (1) exists.
 - b. What is meant by the least squares solution to (1)?
(give a precise definition, not a method to compute it).
 - c. Define three algorithms to compute a least squares solution to (1). Compare their computational cost and accuracy (especially when A is ill-conditioned or rank deficient).
2. Compute a linear least squares fit to the data generated by $f(x) = \sqrt{x}$:
 - a. on a discrete set of three points, $x = \{.25, .49, 1\}$,
 - b. on a continuous interval $x = [0, 1]$.Then give formulas for norms of the two residuals (without actually computing them).

Screening Exam in Numerical Analysis, Sept. 12, 2005
Do all 4 problems

1. Linear equations

Consider a linear system

$$Ax = b \quad (1)$$

where A is a square matrix of order n and vectors x and b have dimension n .

- a. Give conditions under which a solution to (1) exists.
- b. Give formulas (using matrix notation) that define the most commonly used algorithms to compute a solution to (1). Compare their computational cost and accuracy.
- (c). Define the row and column rank of a matrix. How are they related to part (a)? Determine the rank of the following 3×3 matrix A :

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

Find an integer-valued vector b for this matrix

- (i) for which $Ax = b$ has no solutions,
- (ii) for which $Ax = b$ has infinitely many solutions, and give the explicit form of these solutions (depending on some parameter c).

Problem 2. Iterative Solution of Linear Equations

Let A be a real non-singular $n \times n$ matrix and b a real n -dimensional vector. Consider the equation $Ax = b$. Let $A = M - N$ be a splitting of A such that M is non-singular.

(a) Use the splitting to write the equation for an iteration that computes a sequence of vectors, $x^{(m)}$. If the sequence converges, prove that the limit is the solution of $Ax = b$.

(b) Define $Q = M^{-1}N$. Give a sufficient condition on a norm of Q for the convergence of $x^{(m)}$. Prove convergence using this condition. Define the rate of convergence.

(c) Give the splitting for the Jacobi iteration. State a property of A in terms of its diagonal elements that implies the sufficient condition in (b) holds for the Jacobi iteration. Discuss how to parallelize the Jacobi iteration on a cluster of computers.

(d) Give the splitting for the Gauss-Seidel iteration and state a property of A in terms of its elements that is sufficient for convergence. Prove it. Define the SOR family of relaxation iterations with parameter ω which yields the Gauss-Seidel iteration when $\omega = 1$. Discuss how to choose ω to minimize the spectral radius of Q_ω .

3. Eigenvalue problems

Suppose that A is a matrix of size $m \times m$.

(a) Prove Hadamard's circle theorem on the location of the eigenvalues of A :
Theorem. Let P_s be the sum of the moduli of the elements along the s -th row excluding the diagonal element $a(s,s)$. Then each eigenvalue of A lies inside or on the boundary of at least one of the circles $|z - a(s,s)| = P_s$.

(b) Give estimates based on Hadamard's circle theorem for the eigenvalues $z(1)$, $z(2)$, $z(3)$ of the matrix

$$\begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & e \\ 0 & e & 1 \end{pmatrix}$$

where $|e| < 1$.

(c) Establish the tighter bound $|z(3) - 1| \leq e^2$ on the smallest eigenvalue $z(3)$. (Hint: Find a suitable diagonal matrix similarity transformation.)

(d) Explain the QR algorithm with shifts. Indicate the order of convergence of this algorithm and under what conditions the order of convergence is achieved.

Problem 4 . Least Squares.

Let A be an $m \times n$ real matrix.

(a) Show that A is a linear operator mapping the real n -dimensional vector space, V^n , into the real m -dimensional vector space V^m . Both spaces have an inner-product $\langle u, v \rangle$. Show that the transpose of A is a linear operator A^* mapping V^m into V^n and satisfying the relation

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \text{ for all } x \in V^n \text{ and } y \in V^m.$$

(b) Define the l_2 norms. Define the null space N_A of A and the range R_A . The orthogonal complement of N_A is the subspace $N_A^\perp = \{x : \langle x, u \rangle = 0 \text{ for all } u \in N_A\}$. Prove that A maps N_A^\perp onto R_A in a 1:1 way, so that $\dim(N_A^\perp) = \text{rank}(A)$ and $V^n = N_A \oplus N_A^\perp$.

(c) Let $f(x) = \|Ax - b\|^2$, where $x \in V^n$ and $b \in V^m$.

Prove that f has a global minimum. (Hint: To minimize $f(x)$ first find the vector in $R(A)$ ($= R_A$) which is closest to b , where $b = b_{R(A)} + b_{R(A)^\perp}$. Then there exists x^* such that $Ax^* = b_{R(A)}$.) Is the minimum unique? Explain your answer.

(d) The *pseudoinverse* of A is a linear operator A^\dagger on V^m to V^n satisfying the Moore-Penrose conditions :

$$\text{(Gen)} \quad A A^\dagger A = A \text{ and } A^\dagger A A^\dagger = A^\dagger;$$

$$\text{(Sym)} \quad (A A^\dagger)^* = A A^\dagger \text{ and } (A^\dagger A)^* = A^\dagger A.$$

Assume A^\dagger exists. Show that $A^{\dagger\dagger} = A$.

It can be proved that $R(A^\dagger) = N(A)^\perp$ and $N(A^\dagger) = R(A)^\perp$. Use this to show that $A A^\dagger = P_{R(A)}$, where P denotes the orthogonal projection.

Then prove the following Theorem:

Theorem . $A^\dagger b$ is a solution of the least squares problem, that is,

$$\|A A^\dagger b - b\| \leq \|Ax - b\| \text{ for all } x \in V^n.$$

(Hint: $AA^\dagger b = b_{R(A)}$.)

(e) Describe a method for computing A^\dagger . (Hint: State the Singular Values Decomposition (SVD) Theorem or describe the QR method.)

Screening Exam in Numerical Analysis, Spring 2006

1. Iterative methods for solving (*) $Au = f$

1. Consider a square matrix $A \in R^{n \times n}$ such that $A_{ij} = \begin{cases} 2 & \text{for } i = j \\ -1 & \text{for } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$

The eigenvalues of A are known to be $\lambda_i(A) = 2 + 2\cos(\frac{\pi i}{n+1})$, $i=1, \dots, n$.

Compute the following estimates as a function of n (you may use the big-Oh notation, i.e. $O(n)$, $O(n^2)$, etc. in the final answer):

- The condition number of A , $\kappa(A)$,
 - Spectral radius, ρ_J of the Jacobi iteration matrix for (*),
 - Number of Jacobi iterations required.
2. State the assumptions and present the conclusions of the theorem for finding the optimal SOR parameter ω_{opt} (and the corresponding spectral radius, $\rho_{\omega_{opt}}$ of the SOR iteration matrix).

Compute the following estimates for the matrix in (1):

- Optimal SOR parameter, ω_{opt} and the spectral radius, $\rho_{\omega_{opt}}$,
- Number of optimal SOR iterations required.

Problem 2. Solution of Linear Equations by Direct Methods

Given a non-singular $n \times n$ real matrix $A = (a_{ij})$ and an n -dimensional vector b .

(a) Write the equations for an algorithm to compute the LU decomposition of A without pivoting. (Assume that the LU factors exist.) How many arithmetic operations are required?

(b) Write the equations for forward and back substitution to solve $Ax = b$, given that $A = LU$. How many operations are required?

(c) Define the condition number, κ , of A . Use κ to specify a formula for an estimate of rounding error in solving the system $Ax = b$. Define what is meant by an ill-conditioned system. What value of κ would suggest ill-conditioning?

Numerical Analysis Exam Feb 2006

3. (Least squares) Let A be a linear operator mapping a real n -dimensional vector space, V^n , into a real m -dimensional vector space V^m . Both spaces have an inner-product $\langle u, v \rangle$. The *adjoint* of A is a linear operator A^* mapping V^m into V^n and satisfying the relation $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x \in V^n$ and $y \in V^m$.

(a) Let bases be chosen and A represented by an $m \times n$ matrix relative to these bases. With the usual definition of $\langle u, v \rangle$, show that A^* is represented by the transpose of the matrix.

(b) The *pseudoinverse* of A is a linear operator A^\dagger on V^m to V^n satisfying the Moore-Penrose conditions :

(Gen) $AA^\dagger A = A$ and $A^\dagger A A^\dagger = A^\dagger$; (Sym) $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$.

Assume A^\dagger exists. It can be proved that $AA^\dagger = P_{R(A)}$ (projection onto the range)

Use this to prove the following Theorem.

Theorem . $A^\dagger b$ is a solution of the least squares problem, that is,

$$\|AA^\dagger b - b\| \leq \|Ax - b\| \text{ for all } x \in V^n.$$

(Hint: $AA^\dagger b = b_{R(A)}$, where $b = b_{R(A)} + b_{R(A)^\perp}$. For any x ,

$$\|Ax - b\|^2 = \|Ax - b_{R(A)}\|^2 + \|b_{R(A)^\perp}\|^2 \geq \|b_{R(A)^\perp}\|^2. \text{ Justify each step.})$$

(c) State the Singular Values Decomposition (SVD) Theorem for A^\dagger .

4. Eigenvalue problems

Suppose that A is a matrix of order $m \times m$.

1. Give an argument to support the idea that an algorithm for finding eigenvalues needs to be an iterative one.
2. State the QR algorithm with shifts. Indicate its order of convergence.
3. Explain how Householder reflectors and Hessenberg matrices are used in eigenvalue computations (QR algorithm).

Screening Exam on Numerical Analysis – Fall 2006

1. **Gaussian-Seidel Method**

Consider the $n \times n$ matrix A_n defined by

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

(a) Show that the vectors \vec{v}_k^n defined by

$$\vec{v}_k^n = \begin{pmatrix} \sin \frac{k\pi}{n+1} \\ \sin \frac{2k\pi}{n+1} \\ \vdots \\ \sin \frac{nk\pi}{n+1} \end{pmatrix}$$

are eigenvectors of matrix A_n . Find the associated eigenvalues.

(b) Show that Gauss Seidel method converges for solving equation $A_n x = b$.

(c) Find the limit of the condition number of A_n as n tends toward infinity.

2. **Least square**

Suppose you are given the singular value decomposition (SVD) of a matrix A .

(a) Explain how to use this SVD of A to obtain a simple solution to the least square problem

$$\min_x \|Ax - b\|_2$$

(b) Using your work in part a), give an algorithm to solve the least square problem. Show also, that your algorithm can be used to obtain the minimum norm solution.

3. Numerical Integral

(a) A Legendre polynomial $L(x)$ of degree n satisfies

- $\int_{-1}^1 L(x)p(x)dx = 0$ for any polynomial $p(x)$ with degree less than n ,
- $L(1) = 1$

Find $L(x)$ of degree 3.

(b) Show that if f and g are polynomials of degree less than n , if $x_i, i = 1, 2, \dots, n$ are the roots of Legendre polynomial with degree n , and if

$$\gamma_i = \int_{-1}^1 l_i(x)dx$$

with

$$l_i(x) = \prod_{k=1, k \neq i}^n \frac{x - x_k}{x_i - x_k}, i = 1, 2, \dots, n$$

then

$$\int_{-1}^1 f(x)g(x)dx = \sum_{i=1}^n \gamma_i f(x_i)g(x_i).$$

(Hint: one can write $f(x)g(x) = L(x)q(x) + r(x)$, where $L(x)$ is Legendre polynomial with degree n .)

4. Linear Equation

We say a vector $X = (x_1, \dots, x_n)'$ is an *oscillation* vector, if $x_i x_{i+1} < 0$ for all $1 \leq i \leq n - 1$. Let $n \geq 2$ be an integer and t be a real number with $0 < t < 1$. Let A_n be $n \times n$ tridiagonal matrix with

$$A_n = \begin{pmatrix} t & 1 & & & \\ -1-t & t & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1-t & t & 1 \\ & & & -1-t & t \end{pmatrix}.$$

$X = (x_1, \dots, x_n)'$ is the solution of linear system $A_n X = e_n$, where $e_n = (0, \dots, 0, 1)'$ is a n -dimensional unit column vector.

- Prove that X is an oscillation vector, when $n = 2$.
- Prove that X is an oscillation vector for all $n \geq 2$.

(Remark: This partially explains the typical oscillation behavior of Galerkin finite element solution for BVP, since error vector satisfies similar linear equation.)

Screening Exam in Numerical Analysis, Spring 2007

There are questions on 4 chapters covered in M502a.

I. Linear systems

1. Give a definition of a Symmetric Positive Definite (SPD) matrix $A \in R^{n \times n}$.
2. Show that no pivoting is necessary during Gaussian Elimination of a SPD matrix.

3. Show that matrix $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ is SPD.

II. Iterative solutions.

Consider solving a system $Ax = b$, where $x, b \in R^n$, $A \in R^{n \times n}$ using semi-direct iterations, i.e. splitting the system into $Mx = (M - A)x + b$ and iterating as $Mx_{k+1} = (M - A)x_k + b$, or $x_{k+1} = Qx_k + c$, where the iteration matrix $Q = I - M^{-1}A$.

1. Define M for the Jacobi, Gauss-Seidel and SOR iterations.
2. Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
 - a. Find Q for the Jacobi, Gauss-Seidel and SOR iterations.
 - b. Find the optimal overrelaxation parameter ω for this Q_{SOR} .
 - c. Compute spectral radius of all three Q matrices and compare their rate of convergence.

III. Eigenvalue problems

1. Prove the *Gerschgorin's theorem*: All the eigenvalues of the matrix $A \in \mathbb{C}^{n \times n}$ lie in the union of the Gerschgorin disks in the complex plane

$$\mathcal{D}_i = \{z : |z - a_{ii}| \leq r_i\}, \quad r_i = \sum_{j \neq i, j=1}^n |a_{ij}|, i = 1, 2, \dots, n.$$

Moreover, if the union \mathcal{M} of k Gerschgorin disks \mathcal{D}_i is disjoint from the remaining disks, then \mathcal{M} contains precisely k eigenvalues of A .

2. Consider the matrix

$$B = \begin{pmatrix} -5 & 1 & 0 & 0 \\ a & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

with $1 \leq a \leq 3$. Show that the dominant eigenvalue of B is real.

IV. Least squares

1. Given the matrix $A \in \mathbb{R}^{m \times n}$ with rank n . Show that $A^T A$ is symmetric positive definite.

2. Suppose $b \in \mathbb{R}^m$. Show that $x = (A^T A)^{-1} A^T b$ minimizes $\|b - Ax\|_2$.

3. If the matrix is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Find pseudoinverse A^\dagger .

Screening Exam in Numerical Analysis – Fall 2007

Name _____

Solve any three out of the following four problems (do not attempt more than three).

1. linear equations

a) Define the condition number of a matrix A and explain briefly how it is related to the numerical solving of the system $Ax = b$. Prove that $\|A\| \cdot \|A^{-1}\| \geq 1$ for any operator norm $\|\cdot\|$, provided that A^{-1} exists.

b) Let $A \in R^{n \times n}$ be a nonsingular matrix. Show that the condition number $\kappa(A)$ satisfies

$$\kappa(A) \geq \frac{\|A\|}{\|B - A\|}$$

for any singular matrix $B \in R^{n \times n}$.

c) Suppose $0 < |\varepsilon| < 1$. Given matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{bmatrix},$$

Show that

$$\kappa_{\infty}(A) \geq \frac{3}{2\varepsilon},$$

where κ_{∞} is the condition number with respect to infinity norm.

2. iterative methods

a) Let $B \in R^{n \times n}$ and $c \in R^n$. Denote the spectral radius by $\rho(B)$.

Suppose $\rho(B) < 1$, then show that

$$\lim_{n \rightarrow \infty} B^n = 0.$$

b) Prove that A stationary iterative method

$$x^{(n+1)} = Bx^{(n)} + c$$

is convergent to the unique solution of $(I - B)x = c$ for all initial vector $x^{(0)}$ provided that $\rho(B) < 1$.

c) Consider the iteration of (b). Let $\|B\| \leq \beta < 1$, and $\|x^{(k)} - x^{(k-1)}\| \leq \varepsilon$ for some k . Prove that

$$\|x - x^{(k)}\| \leq \frac{\beta\varepsilon}{1 - \beta}$$

3. eigenproblems

a. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 1 & 9.8 \end{pmatrix}$.

- i. Explain the slow rate of convergence of the power method with A .
 - ii. Choose a suitable shift σ so that power method converges to the largest eigenvalue of A . Explain the improvement of the rate of convergence with $A - \sigma I$.
- b. Show that a product (both HR and RH) of a nonsingular right triangular matrix (R) with a Hessenberg matrix (H) is a Hessenberg matrix.

4. least squares

Consider the following least squares minimization problem: find a minimum norm solution of the functional

$$J(x) = \|Bx - c\|^2,$$

where B is a $n \times m$ real matrix and $c \in \mathbb{R}^m$ is given.

- (a) Write down the minimum norm solution of the least square minimization problem as a solution of system of linear equation of the form $Ax = y$ and determine whether or not a unique solution of $Ax = y$ exists.
- (b) Express the minimum norm solution of the least square minimization problem as a function of the eigenvectors of the matrix A . Justify your answer.

A penalty method approach for solving this problem is to consider an alternative minimization problem of finding x_λ that minimizes

$$J_\lambda(x) = \|Bx - c\|^2 + \lambda\|x\|^2,$$

for $\lambda > 0$.

- (c) Write down the solution of penalty method as a function of the eigenvectors of the matrix A .
- (d) Show that x_λ tends toward the solution x of the original least square minimization problem as λ tends toward infinity.

Screening Exam on Numerical Analysis – Spring 2008

Name _____

1. Linear Equations

a. Let $A \in \mathbb{R}^{n \times n}$. Show that the sum of eigenvalues of A is equal to the sum of the diagonal elements of A .

Hint: compare $\det(A - \lambda I)$ and $p(\lambda) = (\lambda_1 - \lambda) \dots (\lambda_n - \lambda)$.

b. Consider solving $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is non-singular. Derive the estimate for the relative error in the solution x if the right hand side b is perturbed by δ .

c. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 + \epsilon \end{pmatrix}, \epsilon > 0$.

How small ϵ should be for you to call the matrix ill-conditioned?

2. Least Squares

a. Prove the following Theorem:

If $A = QR$ with $Q^T Q = I$, then the least squares solution to $Ax = b$ is $x = R^{-1} Q^T b$,

where A is a $n \times m$ matrix.

b. Compute QR factorization of $A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

3. Eigenvalue Problems

a. Let $B \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{n \times m}$ be two given matrices. Prove that the nonzero eigenvalues of BC and CB are the same.

b. Suppose $n \geq 1$ and $T \in \mathbb{R}^{(2n+1) \times (2n+1)}$ is a tridiagonal matrix of the form

$$T = \begin{bmatrix} a_1 & b_2 & & & & \\ & b_2 & a_2 & b_3 & & \\ & & b_3 & \ddots & \ddots & \\ & & & \ddots & a_{2n} & b_{2n+1} \\ & & & & b_{2n+1} & a_{2n+1} \end{bmatrix}$$

$b_i \neq 0$ for all i but $b_{n+1} = 0$. Show that there exists at least one eigenvalue with multiplicity one. (Hint: Study the multiplicity of eigenvalues of $T_1 \in \mathbb{R}^{n \times n}$ and

$T_2 \in \mathbb{R}^{(n+1) \times (n+1)}$, where $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$.)

c. Perform one step of a QR algorithm without shift on $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & 0 \end{bmatrix}$

4. Iterative Methods

Consider $A \in \mathbb{R}^{n \times n}$ to be strictly positive definite, that is, $\xi^T A \xi > 0$ for all nonzero $\xi \in \mathbb{R}^n$. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are eigenvalues and spectral radius is $\rho = \max_i |\lambda_i(A)|$.

a. Show that $\rho \leq \|A\|$ for any consistent matrix norm $\|\cdot\|$.

b. We use the Richardson iteration formula to solve $Ax = b$:

$$x^{(k+1)} = x^{(k)} + \omega(b - Ax^{(k)}), \quad k = 0, 1, 2, \dots \quad (1)$$

Show that (1) is convergent for any $0 < \omega < 2/\rho$.

c. Compare two iterations of (1), by taking $\omega_1 = 1/\rho$ and $\omega_2 = 1/(\lambda_1 + \lambda_n)$. Which one has faster convergence rate? (Hint: compare spectral of the iteration matrix for these two cases).

Screening Exam in Numerical Analysis – Fall 2008

Linear Algebra

1. Perform LU factorization on Hilbert matrix $H_3 = [h_{ij}]_{1 \leq i, j \leq 3}$, with elements

$$h_{ij} = \frac{1}{i+j-1}.$$

2. Let $A \in \mathbb{R}^{n \times n}$ have LU factorization, and $P \in \mathbb{R}^{n \times n}$ be given by $P = (e_n, e_{n-1}, \dots, e_1)$, where e_i is unit vector. Prove that PAP has UL factorization, that is, there exists upper triangular U and lower triangular L satisfying $PAP = UL$.
3. Let $B = [b_{ij}]_{1 \leq i, j \leq n} \in \mathbb{R}^{n \times n}$ be symmetric positive definite. Show that for any $1 \leq i, j, k \leq n$

$$b_{ij} + b_{jk} + b_{ki} \leq b_{ii} + b_{jj} + b_{kk}.$$

Least squares

1. Let $A = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find orthonormal matrix $Q \in \mathbb{R}^{3 \times 2}$ and upper-triangular matrix $R \in \mathbb{R}^{2 \times 2}$, such that $A = QR$.
2. Let $b = (2, -1, 1)^T$. Find $x \in \mathbb{R}^{2 \times 1}$, which minimizes $\|Ax - b\|_2$.
3. Prove Hadamard's determinant inequality:
If $A = (a_1, a_2, \dots, a_n) \in \mathbb{R}^{n \times n}$, then

$$|\det(A)| \leq \prod_{j=1}^n \|a_j\|_2,$$

with equality only if $A^T A$ is diagonal matrix or A has a zero column.
(Hint: Consider QR factorization $A = QR$.)

Iterative Methods

1. Consider solving $Au = f$, where $A \in R^{n \times n}$ is consistently ordered.
 - a. Give the matrix form of Jacobi, Gauss-Seidel and SOR iterations.
 - b. If the eigenvalues of the Jacobi iteration matrix, Q_J are $\lambda_i(Q_J) = \cos(\frac{\pi i}{n+1})$, $i=1, \dots, n$, what is the optimal over-relaxation parameter ω_{opt} ?
2. Consider solving $Au = f$, where $A \in R^{n \times n}$ and $A = A^T$.
 - a. Define the conjugate gradient method.
 - b. Give the estimate of its rate of convergence.
 - c. Compute estimate of the rate of convergence if the eigenvalues of A are $\lambda_i(A) = 2 + 2\cos(\frac{\pi i}{n+1})$, $i=1, \dots, n$.

Eigenvalue Problems.

1. Show that if X is a unitary matrix, and the first column of X is an eigenvector of A associated with eigenvalue λ , then

$$X^*AX = \begin{bmatrix} \lambda & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

2. Consider the matrix

$$A = \begin{bmatrix} -2 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -2 & 3 \end{bmatrix},$$

with an eigenvalue $\lambda = 2$ and corresponding eigenvector $x = [1, 2, 2]^T$.

Construct a Householder matrix H such that

$$HAH^* = \begin{bmatrix} 2 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

Name _____

1. Linear systems

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{pmatrix}$.

- a. Compute LU decomposition of A , i.e. find such L and U that $A = LU$.
- b. Show that A is a SPD matrix. Then compute Cholesky decomposition of A , i.e. find such L that $A = LL^T$.

2. Least Squares

Consider $Ax = b$, where $A \in R^{m \times n}$, $b \in R^m$. A minimum norm solution of the least squares problem is a vector $x \in R^n$ with minimum Euclidian norm that minimizes $\|Ax - b\|_2$.

a. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Find range and null space of A^T , find least squares solution to $Ax = b$, and find minimum norm solution: $\min \|x\|_2$.

- b. Show that a vector that minimizes $\|Ax - b\|_2$ is a minimum norm solution if and only if x is in the range of A^T .

3. Eigenvalue problems

- a. Describe the QR iteration algorithm, present steps of efficient implementation, indicate why the method is numerically stable.
- b. Verify that the eigenvalues are preserved in each step of shifted QR iteration algorithm.
- c. What choice of the rotation angle θ will make A_0 tridiagonal?

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} = U^{-1}AU,$$

where $s = \sin \theta$, $c = \cos \theta$, $|\theta| \leq \pi/2$.

4. Iterative methods

- a. Consider the iterative method $x_{k+1} = -2x_k + b$ to solve the linear system $3Ix = b$, where I is $n \times n$ identity matrix.

For what values of the initial vectors x_0 the iteration converges? What is the spectral radius of iteration matrix?

- b. Let A be a $n \times n$ matrix such that $A = (1 + \omega)P - (N + \omega P)$, with $P^{-1}N$ nonsingular and with real eigenvalues $1 > \lambda_1 \geq \dots \geq \lambda_n$.

Find the values $\omega \in \mathbf{R}$ for which the following iterative method

$$(1 + \omega)Px_{k+1} = (N + \omega P)x_k + b,$$

with $k \geq 0$, converges to the solution of $Ax = b$ for every initial vector x_0 .

Determine the values of ω for which the convergence rate is maximum.

Numerical Analysis Screening Exam, Fall 2009

First Name:

Last Name:

PROBLEM 1 (LEAST SQUARES)

1. Define the Least Squares process using QR factorization for solving $Ax = b$, where $A \in \mathbb{R}^{m \times n}$.

2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

Using Householder QR factorization solve $Ax = b$.

PROBLEM 2 (EIGENVALUE PROBLEMS)

1. State the Schur Theorem.
2. Using the Gram-Schmidt process define the orthonormal set $\{q_1, q_2\}$ for set of vectors $\{v_1, v_2\}$.
3. Find the Schur decomposition of matrix $A = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}$.

PROBLEM 3 (ITERATIVE METHODS)

Consider an iterative scheme of the form

$$(rI + H)x_{k+1} = (rI - H)x_k + b,$$

where H is a symmetric positive definite $n \times n$ matrix, $b \in \mathbb{R}^n$ and r is a positive constant.

- (a) Rewrite the iteration in the form $x_{k+1} = Bx_k + c$.
- (b) Show that the sequence x_k converges for any x_0 .
- (c) If the matrix is only non-negative definite, does the sequence still converge for any b and x_0 ?

PROBLEM 4 (LINEAR SYSTEMS)

(1) Let A be any $n \times n$ matrix and $\|\cdot\|$ be any norm on \mathbb{R}^n (Euclidean n -dimensional space). If $\|I - A\| < 1$, then show that A is invertible and derive the estimate

$$\|A^{-1}\| < \frac{1}{1 - \|I - A\|}.$$

(2) An $n \times n$ matrix $A = [a_{i,j}]$ is strictly diagonally dominant if

$$|a_{i,i}| > \sum_{j=1}^n |a_{i,j}|, \text{ for } i = 1, \dots, n.$$

Show that any strictly diagonally dominant matrix A is invertible. (Hint: recall that $\|A\|_\infty = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |a_{i,j}| \right\}$ and write $A = DB$ where D is the diagonal part of A and show that $\|I - B\|_\infty < 1$.)

Numerical Analysis Screening Exam, Spring 2010

First Name:

Last Name:

I. Linear Equations

Consider the singular system

$$Bu = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} u = f.$$

- (a) Find the range $\mathcal{R}(B)$ and null space $\mathcal{N}(B)$ of B .
- (b) State the solvability condition for $Bu = f$.
- (c) Find an example of f for which $Bu = f$ has *no* solutions.
- (d) Find an example of f for which $Bu = f$ has *infinitely many* solutions, and find the explicit form of this solution (depending on a parameter c).

II. Least Squares Problem

Consider the matrix A and vector b given below

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -2 \\ 1 & 5 & 6 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

- (a) Using the Householder transformation matrix method to find a QR -decomposition of the matrix A and a solution to the problem

$$\min \|Ax - b\|_2^2.$$

- (b) Find the minimum norm solution of the above least squares minimization problem.

III. Eigenvalue Problems

- (a) First consider the following matrix

$$B = \begin{pmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}.$$

Find any real eigenvalue of B and any associated eigenvector.

- (b) Now let A be any $n \times n$ matrix. Show that $\det(A) = \prod_{i=1}^n \lambda_i$ where $\lambda_1, \dots, \lambda_n$ are eigenvalues of A . (Hint: Consider characteristic polynomial of A)
- (c) Show that A is singular if and only if $\lambda = 0$ is an eigenvalue of A .

IV. Iterative Methods

- (a) Consider the iterative method:

$$x_{k+1} = Bx_k + c, \quad k = 1, 2, \dots,$$

where B is a $n \times n$ matrix and x_0 and c are arbitrary vectors of \mathbb{R}^n . Define the spectral radius $\rho(B)$ of B and show that x_k converges for all initial vectors x_0 if $\rho(B) < 1$.

- (b) Consider the Richardson iterative scheme

$$x_{k+1} = x_k + \omega(b - Ax_k), \quad k = 1, 2, \dots,$$

where A is a $n \times n$ matrix and ω is a positive number. We assume that all the eigenvalues λ_i of A are real and satisfy $0 < \alpha \leq \lambda_i \leq \beta$ for $i = 1, \dots, n$ and for some positive values α and β . Find the condition on the number ω in terms of α and β such that for any initial vector x_0 , the sequence x_k converges to the solution of $Ax = b$.

- (c) Assuming that $\alpha = \min_{1 \leq i \leq n} \lambda_i$ and $\beta = \max_{1 \leq i \leq n} \lambda_i$ in (b), what value of ω leads to the fastest convergence of the scheme.

Numerical Analysis Screening Exam, Fall 2010

FIRST NAME:

LAST NAME:

STUDENT ID NUMBER:

SIGNATURE:

PROBLEM 1 Let A be an $m \times n$ matrix and b an $m \times 1$ vector. Consider the least squares problem (LS): find x that minimizes $\|Ax - b\|_2^2$.

- (a) Give necessary and sufficient conditions for x to be a solution to (LS).
- (b) When is this solution unique?
- (c) When is the corresponding residual vector $r = Ax - b$ unique?
- (d) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find the solution x to (LS) that also minimizes $\|x\|_2^2$.

Name:

Name: _____

PROBLEM 2. A symmetric $n \times n$ matrix A is called Symmetric Positive Definite (SPD) iff $(Ax, x) > 0$ for all $x \neq 0$.

(a) Assume $n = 2$. Show that if A is SPD then A admits a Cholesky factorization, i.e $A = L \cdot L^T$ where L is a nonsingular lower triangular matrix.

(b) Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

Show that A is SPD and calculate the Cholesky factorization of A .

Name:

Name: _____

PROBLEM 3.

- (a) Why does any eigenvalue solver have to be iterative?
- (b) Present the QR-algorithm to solve eigenvalue problems in some detail and state the corresponding theorem for convergence.
- (c) Let $A_k = \begin{pmatrix} c & s \\ s & 0 \end{pmatrix}$, where $c = \cos \theta$ and $s = \sin \theta$. Compute A_{k+1} using QR-iteration, and show that the off-diagonal elements of A_{k+1} are smaller than those in A_k .

Name:

Names: _____

PROBLEM 4. Consider a matrix A given by

$$A = \begin{pmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{pmatrix}$$

- (a) Verify that $x = [2, 2, 1]^T$ is a solution of $Ax = [1.001, 1.001, 0.001]^T$. Consider the vectors $y = [2.2, 2.2, 1]^T$ and $z = [202, 202, 201]^T$. Verify that $Ax - Ay$ and $Ax - Az$ have the same infinity vector norm.
- (b) Find an estimation of the condition number of A using the results of (a).
- (c) Verify that

$$A^{-1} = \begin{pmatrix} -899 & 900 & 999 \\ -999 & 1000 & 999 \\ -900 & 900 & 1000 \end{pmatrix}.$$

Determine whether or not

$$B = \begin{pmatrix} -900 & 900 & 1000 \\ -1000 & 1000 & 1000 \\ -900 & 900 & 1000 \end{pmatrix}$$

is a sufficiently good approximation of the matrix A^{-1} to be used in an iterative method of the form $x_{k+1} = x_k - BAx_k + Bb$ for finding the solution of $Ax = b$.

(Hint: $B \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$)

Name:

Numerical Analysis Screening Exam, Spring 2011

FIRST NAME:

LAST NAME:

STUDENT ID NUMBER:

SIGNATURE:

PROBLEM 1. (LINEAR EQUATIONS)

- (a) Give a definition of matrix A being positive definite.
- (b)-i State **any** theorem for solving $Ax = b$ with symmetric positive definite (SPD) matrices.
- (b)-ii What are the computational advantages for solving a problem $Ax = b$ with SPD matrices. Be specific.
- (c) Find the largest interval for α so that A is positive definite:

$$A = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}.$$

(Hint: use Gaussian elimination.)

Name: _____

PROBLEM 2. (EIGENVALUE PROBLEMS)

- (a) Let $A = [a_{ij}]$ be an $n \times n$ matrix. Prove Gerschgorin's Theorem which states the following: For $i = 1, \dots, n$ let $R_i = \sum_{j=1, j \neq i}^n |a_{ij}|$. Every eigenvalue of A falls within one of the closed discs in the complex plane with center at a_{ii} and radius R_i . (Hint. Let

$$Ax = \lambda x, \tag{1}$$

and assume the largest component of x in absolute value is x_k . Consider the k -equation of (1).)

- (b) Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that A is positive definite. (Hint. Use Gerschgorin's Theorem to show that A is positive semi-definite. Then consider the equation $Ax = 0$, where the first component of x equals 1.)

- (c) Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Find the eigenvalues of A and verify that Gerschgorin's Theorem holds.

Name: _____

PROBLEM 3. (ITERATIVE METHODS)

- (a) Let L be a strictly lower triangular and U be a strictly upper triangular $n \times n$ matrices. For any positive number ω define

$$B_\omega = (1 - \omega L)^{-1} [(1 - \omega)I + \omega U].$$

Show that $\rho(B_\omega) \geq |\omega - 1|$ where $\rho(B_\omega)$ is the spectral radius of B_ω . (Hint. Calculate $\det(B_\omega)$ directly and by the product of eigenvalues and compare.)

- (b) Consider solving $Ax = b$ where A is an $n \times n$ matrix and $x, b \in \mathbb{R}^n$.
- Give the matrix form of the SOR iterative method.
 - Using the result in question (a) above, show that the SOR method with parameter ω can only converge for $0 < \omega < 2$.
- (c) Consider the equation $Ax = b$, where $x \in \mathbb{R}^3$ and

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Can the SOR method be applied to this equation? Justify your answer.

Names: _____

PROBLEM 4. (LEAST SQUARES PROBLEM)

- (a) Let $A \in R^{m \times n}$. Give a detailed description of the SVD of A and present briefly an algorithm to solve an overdetermined system using SVD.
- (b) What are the advantages and disadvantages of using SVD for overdetermined systems.
- (c) Consider the overdetermined system $Au = b$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Find the family of least squares solutions using **any** method that is easy for hand calculation, and finally find the minimum-norm solution to the least squares problems.