Screening Exam on Numerical Analysis – Fall 2006

1. **Gaussian-Seidel Method**

   Consider the $n \times n$ matrix A_n defined by

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

   (a) Show that the vectors $\vec{v}_k^n$ defined by

$$\vec{v}_k^n = \begin{pmatrix} \sin\frac{k\pi}{n+1} \\ \sin\frac{2k\pi}{n+1} \\ \vdots \\ \sin\frac{nk\pi}{n+1} \end{pmatrix}$$

   are eigenvectors of matrix A_n. Find the associated eigenvalues.

   (b) Show that Gauss Seidel method converges for solving equation $A_n x = b$.

   (c) Find the limit of the condition number of A_n as n tends toward infinity.

2. **Least square**

   Suppose you are given the singular value decomposition (SVD) of a matrix A.

   (a) Explain how to use this SVD of A to obtain a simple solution to the least square problem

$$\min_x \|Ax - b\|_2$$

   (b) Using your work in part a), give an algorithm to solve the least square problem. Show also, that your algorithm can be used to obtain the minimum norm solution.

3. **Numerical Integral**

   (a) A Legendre polynomial $L(x)$ of degree n satisfies

   - $\int_{-1}^{1} L(x)p(x)dx = 0$ for any polynomial $p(x)$ with degree less than n,
   - $L(1) = 1$

   Find $L(x)$ of degree 3.

   (b) Show that if f and g are polynomials of degree less than n, if $x_i, i = 1, 2, \ldots, n$ are the roots of Legendre polynomial with degree n, and if

   $$\gamma_i = \int_{-1}^{1} l_i(x)dx$$

   with

   $$l_i(x) = \prod_{k=1,k\neq i}^{n} \frac{x - x_k}{x_i - x_k}, i = 1, 2, \ldots, n$$

   then

   $$\int_{-1}^{1} f(x)g(x)dx = \sum_{i=1}^{n} \gamma_i f(x_i)g(x_i).$$

   (Hint: one can write $f(x)g(x) = L(x)q(x) + r(x)$, where $L(x)$ is Legendre polynomial with degree n.)

4. **Linear Equation**

   We say a vector $X = (x_1, \ldots, x_n)'$ is an *oscillation* vector, if $x_i x_{i+1} < 0$ for all $1 \leq i \leq n-1$. Let $n \geq 2$ be an integer and t be a real number with $0 < t < 1$. Let A_n be $n \times n$ tridiagonal matrix with

   $$A_n = \begin{pmatrix} t & 1 & & & \\ -1-t & t & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1-t & t & 1 \\ & & & -1-t & t \end{pmatrix}.$$

   $X = (x_1, \ldots, x_n)'$ is the solution of linear system $A_n X = e_n$, where $e_n = (0, \ldots, 0, 1)'$ is a n-dimensional unit column vector.

   (a) Prove that X is an oscillation vector, when $n = 2$.

   (b) Prove that X is an oscillation vector for all $n \geq 2$.

   (Remark: This partially explains the typical oscillation behavior of Galerkin finite element solution for BVP, since error vector satisfies similar linear equation.)