Geometry/Topology Qualifying Exam

September 2006

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. Let M, N be compact oriented manifolds of dimension n (without boundary), and let $f : M \to N$ be a differentiable map. Prove that, if the induced homomorphism $f^* : H^n_{dR}(N;\mathbb{R}) \to H^n_{dR}(M;\mathbb{R})$ between de Rham cohomology groups is surjective, then f is surjective.

2. Let D^2 be the closed unit disk in the complex plane $\mathbb{C}$, bounded by the unit circle S^1. Consider the 2-dimensional torus $T^2 = S^1 \times S^1$ and two copies D_1 and D_2 of D^2. For two integers p, q, let X_{pq} be the quotient space of the disjoint union
$$T^2 \sqcup D_1 \sqcup D_2$$
by the equivalence relation that identifies each point $e^{i\theta}$ in the boundary of D_1 to $(e^{ip\theta}, 1) \in S^1 \times S^1$, and identifies each point $e^{i\phi}$ in the boundary of D_2 to $(1, e^{iq\phi}) \in S^1 \times S^1$. Compute the fundamental group of X_{pq}.

3. Prove that any two continuous maps $f, g : X \to S^1$ from a simply-connected space X to the circle S^1 are homotopic.

4. Calculate the relative homology groups $H_*(S^1 \times D^2, S^1 \times \partial D^2)$, where D^2 denotes the 2-dimensional closed disk and S^1 is the circle.

5. Let M be a compact oriented n-manifold with $H^1_{dR}(M;\mathbb{R}) = 0$ and let $f : M \to T^n$ be a smooth map. Show that the degree of f is equal to 0. (Possible hint: Write $T^n = S^1 \times \cdots \times S^1$; if θ_i is the angular coordinate for the i-th factor S^1, then $d\theta_1 \wedge \cdots \wedge d\theta_n$ is a volume form for T^n.)

6. Recall that the *rank* of a matrix is the dimension of the span of its row vectors. Show that the space of all 2×3 matrices of rank 1 forms a smooth manifold.

7. Consider the group SO(3) of orientation-preserving isometries of the 2-dimensional sphere S^2. Namely, SO(3) consists of all rotations of $\mathbb{R}^3$ whose axis passes through the origin or, equivalently of all 3×3 matrices A such that $AA^t = \mathrm{Id}$ and $\det(A) = 1$. Prove that, if ω is a 1-form (not necessarily closed) on S^2 such that $\phi^*(\omega) = \omega$ for every $\phi \in \mathrm{SO}(3)$, then $\omega = 0$.