1. Let $X_1, \dots, X_n$ be i.i.d. with Poisson distribution $P(\lambda)$ where $\lambda \in (0, \infty)$, that is, for all $i = 1, \dots, n$ that $P(X_i = k) = e^{-\lambda}\lambda^k/k!, k = 0, 1, \dots$ and are independent.

   (a) Show that this family of distributions has the monotone likelihood ratio property with respect to an appropriately chosen statistic $T(X_1, \dots, X_n)$.

   (b) For the statistical model described, give an example of a hypothesis testing problem $H_0 : \theta \in \Theta_0,\ H_a : \theta \in \Theta_a$ where Θ_0, Θ_a are two subsets of $(0, \infty)$ that satisfy $\Theta_0 \cap \Theta_a = \emptyset,\ \Theta_0 \cup \Theta_a = (0, \infty)$, that admits a uniformly most powerful test of any size $\alpha \in [0, 1]$. Justify your answer.

   (c) For the statistical model described above, give an example of a hypothesis testing problem $H_0 : \theta \in \Theta_0,\ H_a : \theta \in \Theta_a$, with $\Theta_0,\ \Theta_a$ satisfying the same properties as in part (b), that **does not** admit a uniformly most powerful test of given size $\alpha \in (0, 1)$. Justify your answer.

2. Let $X = (X_1, \dots, X_n)$ be a random sample of size n from a family of probability densities $\{f_\theta \colon \theta \in \mathbb{R}\}$, so that $f_\theta \colon \mathbb{R}^n \to (0, \infty)$ for any $\theta \in \mathbb{R}$, and $X_1, \dots, X_n$ are i.i.d. Fix $\theta_0 \in \mathbb{R}$. Suppose we test the hypothesis H_0 that $\{\theta = \theta_0\}$ versus the alternative $\{\theta \neq \theta_0\}$. Let $Y = Y_n$ denote the MLE of θ, and assume that under P_{θ_0} that

$$Y_n \to_p \theta_0 \quad \text{and} \quad \sqrt{n}(Y - \theta_0) \to_d \mathcal{N}(0, I_{X_1}^{-1})$$

   where $\to_p$ and $\to_d$ denote convergence in probability and in distribution respectively, $\mathcal{N}$ denote the normal distribution and I_X the Fisher information, which we assume exists.

   Let

$$\lambda(X) := \frac{\sup_{\theta \in \mathbb{R}} f_\theta(X)}{f_{\theta_0}(X)}$$

   denote the generalized likelihood ratio statistic. If H_0 is true, we will ask you to show that $-2 \log \lambda(X)$ converges in distribution as $n \to \infty$ to a chi-squared random variable with one degree of freedom, under some additional assumptions.

   Fix $x \in \mathbb{R}^n$ and denote $\ell_n(\theta) := \log f_\theta(x)$. Below you may assume whatever smoothness conditions you require for your argument, but please note them when they are applied, in the form of, say: here we assume that a second order Taylor expansion for $\ell_n(\theta)$ holds with appropriate form of remainder.

   (a) Using Taylor series, show that

$$-2 \log \lambda(X) = -\ell_n''(\widehat{Y})(\theta_0 - Y)^2$$

   where $\widehat{Y}$ is some point in an interval with endpoints Y and θ_0.

   (b) Using the weak law of large numbers, show that, for any $\theta \in \mathbb{R}$, $\frac{1}{n}\ell_n''(\theta)$ converges in probability to the constant $I_{X_1}(\theta)$ as $n \to \infty$, and that the same conclusion holds for $\frac{1}{n}\ell_n''(\widehat{Y})$.

   (c) Combining the above observations, conclude that $-2 \log \lambda(X)$ converges in distribution to a chi-squared random variable with one degree of freedom.